**MATH 110**
**EXAM #2**

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1.

Please compute the explicit solution $u(x,t)$ to the following problems:

a) The Dirichlet wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \ , & 0 \leqslant x \leqslant \pi \ , \\ u(0,t) &= u(\pi,t) \ = \ 0 \ , \\ u(x,0) &= \sin(x) - 3\sin(4x) \ , \\ u_t(x,0) &= 2\sin(2x) + 7\sin(9x) \ . \end{aligned}$$

b) The Neumann heat equation:

$$\begin{aligned} u_t &= k u_{xx} \ , & 0 \leqslant x \leqslant 1 \ , \\ u_x(0,t) &= u_x(1,t) \ = \ 0 \ , \\ u(x,0) &= 2 + 2\cos(\pi x) - 5\cos(3\pi x) \ . \end{aligned}$$

What is the steady-state temperature $T_\infty = \lim_{t\to\infty} u(x,t)$?

## Problem 2.

Consider the two explicit functions:

$$\begin{aligned}\phi(x) &= 2\sin(\pi x) - 4\sin(2\pi x) + 3\sin(4\pi x) - 10\sin(5\pi x) + 5\sin(6\pi x) \ , \\ \psi(x) &= -\sin(\pi x) + 6\sin(2\pi x) - 2\sin(5\pi x) + 3\sin(6\pi x) - \sin(7\pi x) \ .\end{aligned}$$

Please compute the integral:

$$\langle \phi, \psi \rangle \ = \ \int_0^1 \ \phi(x)\psi(x) \ dx \ .$$

(Hint: You may freely quote the values of certain integrals we have covered. The answer can be given quickly, but you need to explain clearly what you are doing. Do *not* use a calculator to get a numerical answer.)

## PROBLEM 3.

Consider the following mixed boundary value problem for the Heat equation:

$$\begin{aligned}
(1) \qquad & u_t = k u_{xx} \,, & 0 \leqslant x \leqslant \ell \,, \\
(2) \qquad & u_x(0,t) = u(\ell,t) = 0 \,, & \\
(3) \qquad & u(x,0) = f(x) \,. &
\end{aligned}$$

Please answer the following:

a) Lets try to solve this via a series expansion. First of all, consider special solutions of the form $u(x,t) = X(x)T(t)$ which lead to the boundary value problem:

$$(4) \qquad -X'' = \lambda X \,, \qquad X'(0) = X(\ell) = 0 \,.$$

Are these boundary conditions symmetric?

b) Solve the ODE (4) by finding the correct values of $\lambda_n$ which make the solution possible (Hint: Just start with $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ and then find conditions on $A$ and $B$ first. This will then determine the values of $\lambda$).

c) Recall that once we have explicit solutions $\{X_n, \lambda_n\}$ to (4), we can write a general solution to (1) with boundary conditions (2) as:

$$u(x,t) = \sum_n A_n e^{-k\lambda_n t} X_n(x) \,.$$

What is the behavior of this solutions as $t \to \infty$? Physically why does this make sense?

d) Suppose that the $X_n$ from part a) are chosen so that $\int_0^\ell X_n^2(x)dx = 1$ (this can always be accomplished by dividing through by a correct constant). Suppose also that the initial data (3) $f(x)$ is such that:

$$\int_0^\ell f(x)X_1(x)\ dx \ = \ 2\ , \quad \int_0^\ell f(x)X_2(x)\ dx \ = \ -3\ , \quad \int_0^\ell f(x)X_3(x)\ dx \ = \ 5\ ,$$

with all other $\int_0^\ell f(x)X_k(x)\ dx = 0$ for $k \neq 1, 2, 3$. Write down a formula for $f(x)$ in terms of the $X_n$. Please explain clearly why this works.