SUBFACTORS FROM BRAIDED C* TENSOR CATEGORIES

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Abstract. We extend subfactor constructions originally defined for unitary braid representations to the setting of braided C* tensor categories. The categorical approach is then used to compute the principal graph of these subfactors. We also determine the dual principal graph for several important cases. Here invertibility of the so-called S-matrix of a subcategory and certain related group actions play an important role.

It was noted by Vaughan Jones that his examples of subfactors gave rise to unitary braid representations. By this we mean representations of the infinite braid group $\mathcal{B}_\infty$ defined by infinitely many generators $\sigma_1, \sigma_2, \ ...$ which satisfy the familiar braid relations. Subsequently, unitary braid representations were used by A. Ocneanu and by H. Wenzl to construct new examples of subfactors; here the subfactor is given by the subgroup $\mathcal{B}_{2,\infty}$ generated by $\sigma_2, \sigma_3, \ldots$. This construction was denoted as the one-sided subfactor construction by J. Erlijman, as opposed to her multi-sided subfactors. Here, for a given integer $s > 1$, the s-sided subfactor is obtained as a suitable inductive limit of the embeddings of the quotients of $\mathcal{B}_n^s = \mathcal{B}_n \times \cdots \times \mathcal{B}_n$ (s times) into $\mathcal{B}_{ns}$ with respect to n. She also computed the indices of these subfactors and their first relative commutants.

The main motivation for this paper was to calculate the higher relative commutants of Erlijman's subfactors. To do this it is convenient to generalize the above mentioned constructions to the setting of a braided C* tensor category $\mathcal{C}$ with only finitely many simple objects up to isomorphism. By definition of such a category, we obtain a unitary representation of $\mathcal{B}_n$ in $\mathrm{End}(X^{\otimes n})$ for any object X in $\mathcal{C}$. The constructions in our paper in the category setting follow closely the above-mentioned braid constructions, and reduce to them in case that $\mathrm{End}(X^{\otimes n})$ is generated by the quotients of $\mathcal{B}_n$ for all $n \in \mathbb{N}$. However, the categorical setting makes it easier to calculate the higher relative commutants, and also contains new nontrivial examples.

The main results of our paper are as follows. We show that the first principal graph is given by the fusion graph of $(\mathcal{C}')^s$, where $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ depending on the tensor powers of X in which the trivial object appears. The fusion graph describes the decomposition of the tensor product of s simple objects of $\mathcal{C}'$ into irreducibles ones; see Theorem 4.6 for details. The situation is more complicated for the dual (or second) principal graph. If a certain matrix depending on the braiding structure, called the S-matrix for the category $\mathcal{C}'$, is invertible, the dual principal graph coincides with the principal graph.

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We do not have a general complete result in the case of a noninvertible S-matrix. It is known that in this case there is a canonical subcategory $\mathcal{T}$ of $\mathcal{C}'$ which is equivalent to the representation category of a finite group G. If G is abelian, we obtain an action of G on the set of irreducible objects of $\mathcal{C}$, which is given by a labeling set Λ. The dual principal graph can now be fairly precisely characterized in terms of the orbits of the action of a group G_1^s on Λ^s, see Theorem 5.9 for details and, for an example, Proposition 6.1.

The basic idea of our paper is that we explicitly construct a number of $\mathcal{A}-\mathcal{B}$ bimodules, with $\{\mathcal{A},\mathcal{B}\} \subset \{\mathcal{N},\mathcal{M}\}$ and with $\mathcal{N} \subset \mathcal{M}$ being our s-sided inclusion. We show that these examples of bimodules are closed under induction and restriction. One deduces from this that the induction-restriction graph for these bimodules must coincide with the principal or dual principal graph under some mild additional assumptions.

Our findings are related to a number of results by different authors. If $s = 2$, our subfactors correspond to the subfactors obtained from the asymptotic inclusion of certain one-sided subfactors. In this case, the orbifold phenomenon for the dual principal graph has first been observed by Ocneanu for the example of the Jones subfactors. Further results have been obtained in [EK2] and [Iz]. In particular, some of our proofs have been inspired by these results. We were also informed by M. Asaeda that, after having heard a talk on this paper, she has obtained an analogue of the s-sided construction under more general conditions.

More or less the same combinatorics as in our paper also appears in the work [X] of Feng Xu on subfactors of type III_1 factors related to disconnected intervals. In spite of the similarity of principal graphs and indices, his construction of these subfactors is completely different from ours and relies on Wassermann's loop group construction, which has not appeared yet for all Lie types.

Here is a more detailed description of the contents of this paper. In the first chapter we review some basic results on bimodules in the type II_1 setting. The second chapter contains definitions concerning braided C^* tensor categories. In the third chapter we present the generalization of previous subfactor constructions to the setting of braided C^* tensor categories, as well as additional technical results. This is used in the following section to construct certain bimodules and compute the principal graph of these subfactors. In the last section we prove the already mentioned results about the dual principal graph. We then discuss examples of our construction including the case of the Jones subfactors.

1. Bimodules

1.1. **Definitions.**

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be a type II_1 factors, and let H be a Hilbert space.

(i) H is a *left $\mathcal{A}$–module* if there exists an action of $\mathcal{A}$ on H determined by a normal unital morphism $\lambda : \mathcal{A} \to B(H)$, where $B(H)$ is the von Neumann algebra of all bounded linear operators on H.

(ii) A *right $\mathcal{B}$–module* H is a left $\mathcal{B}^{\mathrm{opp}}$–module (here, $\mathcal{B}^{\mathrm{opp}}$ denotes the opposite algebra of $\mathcal{B}$).

(iii) H is an $\mathcal{A}$–$\mathcal{B}$ *bimodule* if it is a left $\mathcal{A}$–module, a right $\mathcal{B}$–module, and if the left and right actions intertwine. That is, if $\lambda : \mathcal{A} \to B(H)$ is the left action, and if $\rho : \mathcal{B}^{\mathrm{opp}} \to B(H)$ is the right action, then we must have that $\lambda(a)\rho(b) = \rho(b)\lambda(a)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.

(iv) If H and K are $\mathcal{A}$–$\mathcal{B}$ bimodules, we define the space of *intertwiners*, denoted by $\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(H,K)$, to be the set of linear bounded operators $T : H \to K$ such that they intertwine the actions, that is, such that $T\lambda_H(a) = \lambda_K(a)T$ for all $a \in \mathcal{A}$, and $T\rho_H(b) = \rho_K(b)T$ for all $b \in \mathcal{B}$.

(v) Two $\mathcal{A}$–$\mathcal{B}$ bimodules H and K are *equivalent* or *isomorphic* if there exists a unitary operator in $\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(H,K)$.

Definition 1.2. Let H be an $\mathcal{A}$–$\mathcal{B}$ bimodule with left action λ, and right action ρ. The *inclusion generated by* H is the inclusion of factors given by

$$\lambda(\mathcal{A}) \subset \rho(\mathcal{B})'.$$

The *dual inclusion generated by* H is the inclusion of factors given by

$$\rho(\mathcal{B}) \subset \lambda(\mathcal{A})'.$$

Remark 1.3. Similarly, if we have an inclusion of type II_1-factors $\mathcal{N} \subset \mathcal{M}$, we can make $L^2(\mathcal{M}, tr)$ into an $\mathcal{M} - \mathcal{M}$, $\mathcal{M} - \mathcal{N}$, $\mathcal{N} - \mathcal{M}$ or $\mathcal{N} - \mathcal{N}$-bimodule via usual left and right multiplication. If $\mathcal{N} \subset \mathcal{M}$ is a reducible inclusion, i.e. the relative commutant $\mathcal{N}' \cap \mathcal{M}$ is larger than $\mathbb{C}1$, then we obtain further examples by reducing by projections in the relative commutant. E.g. if $p \in \mathcal{N}' \cap \mathcal{M}$, we obtain the $\mathcal{N} - \mathcal{M}$ bimodule $L^2(p\mathcal{M}, tr)$.

If $\phi_i : \mathcal{M} \to \mathcal{M}$ are endomorphisms for $i = 1, 2$, we can also define an $\mathcal{M} - \mathcal{M}$-bimodule structure on $L^2(\mathcal{M}, tr)$ by perturbing the right and left actions by these endomorphisms, i.e. by defining the action by $m_1.\xi.m_2 = \phi_1(m_1)\xi\phi(m_2)$.

All the examples of bimodules encountered in this paper are of one of these types or tensor products of them.

Definition 1.4. Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be type II_1 factors for $i = 1, 2$. Let H_i be $\mathcal{A}_i$–$\mathcal{B}_i$ bimodules with left actions λ_i and right actions ρ_i, respectively, for $i = 1, 2$. H_1 and H_2 are *(left)-weakly isomorphic* or *(left)-weakly equivalent* if the inclusions generated by H_1 and by H_2 are conjugate, i.e. there exists an isomorphism $\Psi : \rho_1(\mathcal{B}_1)' \to \rho_2(\mathcal{B}_2)'$ such that $\Psi(\lambda_1(\mathcal{A})) = \lambda_2(\mathcal{A})$.

Remark 1.5. 1. In the following we will often suppress the notations λ and ρ for left and right actions if it is clear from the context which algebra acts from which side. We shall also be mostly concerned with (left)-weakly equivalence, so we will usually only refer to it as weak equivalence.

2. With the notations of the last definition, let H_1, H_2 be two equivalent $\mathcal{A}$–$\mathcal{B}$ bimodules. Then it is easy to check that they are also (left)-weakly equivalent. Indeed, let $\Phi : H_1 \to H_2$ be the unitary intertwining the left and right actions. Then the intertwining property implies that $\Phi^{-1}\lambda_1(\mathcal{A}_2)\Phi = \lambda_2(\mathcal{A}_1)$ and $\Phi^{-1}\rho_2(\mathcal{B}_2)\Phi = \rho_1(\mathcal{B}_1)$. But then it also follows that

$\Phi^{-1}\rho_2(\mathcal{B}_2)'\Phi = \rho_1(\mathcal{B}_1)'$, which gives the desired isomorphism between the two inclusions given by H_1 and H_2.

3. The well-known fact that $\Phi^{-1}\rho_2(\mathcal{B}_2)\Phi = \rho_1(\mathcal{B}_1)$ if and only if $\Phi^{-1}\rho_2(\mathcal{B}_2)'\Phi = \rho_1(\mathcal{B}_1)'$ will be repeatedly used in this paper.

Let H_i be $\mathcal{A}$–$\mathcal{B}_i$ bimodules with left actions λ_i and right actions ρ_i, respectively, for $i = 1, 2$. Assume that $\dim_{\mathcal{A}} H_2 \leq \dim_{\mathcal{A}} H_1 < \infty$, where $\dim_{\mathcal{A}} H$ is the Murray-von Neumann dimension of the $\mathcal{A}$-module H.

Lemma 1.6. *H_1 is weakly isomorphic to H_2 if and only if there exists a projection $p \in \mathcal{B}_1$ such that $H_1 p$ and H_2 are isomorphic as $\mathcal{A}$–$\mathcal{B}_2$ bimodules; here $H_1 p = \{\rho_1(p)(x) \ : \ x \in H_1\}$, and the $\mathcal{B}_2$ right module structure on $H_1 p$ is the one induced from $p\mathcal{B}_1 p$ by the spatial isomorphism between $H_1 p$ and H_2.*

Proof. First we shall show the necessity. Since H_1 is weakly isomorphic to H_2, there exists an isomorphism

$$\Psi : \rho_1(\mathcal{B}_1)' \to \rho_2(\mathcal{B}_2)'$$

such that $\Psi(\lambda_1(\mathcal{A})) = \lambda_2(\mathcal{A})$. In particular, $[\rho_1(\mathcal{B}_1)' : \lambda_1(\mathcal{A})] = [\rho_2(\mathcal{B}_2)' : \lambda_2(\mathcal{A})]$. As $\dim_{\mathcal{N}} H = [\mathcal{M} : \mathcal{N}] \dim_{\mathcal{M}} H$ for any inclusions of II$_1$-factors $\mathcal{N} \subset \mathcal{M}$ and any finite-dimensional $\mathcal{M}$ module H, we have

$$1 \geq \alpha := \frac{\dim_{\mathcal{A}} H_2}{\dim_{\mathcal{A}} H_1} = \frac{\dim_{\rho_2(\mathcal{B}_2)'} H_2}{\dim_{\rho_1(\mathcal{B}_1)'} H_1}.$$

Choose a projection $p \in \mathcal{B}_1$ with $tr(p) = \alpha$, so that $\dim_{\mathcal{A}}(H_1 p) = \dim_{\mathcal{A}} H_2$. Then the isomorphism between $p\rho_1(\mathcal{B}_1)'p \cong \rho_1(\mathcal{B}_1)'$ and $\rho_2(\mathcal{B}_2)'$ is spatial, i.e. it is given by conjugation by a unitary intertwiner $\Phi : H_1 p \to H_2$. In particular, we obtain $\Phi p \rho_1(\mathcal{B}_1) p \Phi^{-1} = \rho_2(\mathcal{B}_2)$; this isomorphism between $p\rho_1(\mathcal{B}_1)p$ and $\rho_2(\mathcal{B}_2)$ makes $H_1 p$ into a $\mathcal{B}_2$ right module. By construction, Φ defines an isomorphism between the $\mathcal{A} - \mathcal{B}_2$ modules $H_1 p$ and H_2.

Now, we shall show the sufficiency. Suppose that H_2 and $H_1 p$ are isomorphic as $\mathcal{A}$–$\mathcal{B}_2$ bimodules, where p is a projection in $\mathcal{B}_1$. Observe that the bimodule isomorphism $\Phi : H_1 p \to H_2$ induces a spatial isomorphism between $p\rho_1(\mathcal{B}_1)p$ and $\rho_2(\mathcal{B}_2)$, as described in the last paragraph. This, in turn, induces an isomorphism between their commutants $p\rho_1(\mathcal{B}_1)'p \cong \rho_1(\mathcal{B}_1)'$ and $\rho_2(\mathcal{B}_2)'$. As Φ interwines the $\mathcal{A}$-actions on $H_1 p$ and H_2, this isomorphism maps $p\rho_1(\mathcal{A})p$ to $\rho_2(\mathcal{A})$. $\diamondsuit$

Remark 1.7. There exists an analogous statement of the last lemma for $\mathcal{A}_i$–$\mathcal{B}$ bimodules H_i with left actions λ_i and right actions ρ_i, respectively, for $i = 1, 2$, and with essentially the same proof. We leave the details to the reader.

1.2. **Tensor products.** Tensor products of bimodules have been defined by Connes and Sauvageot. A good review with results for our paper can be found in [Bs].

Proposition 1.8. *Let H_i be $\mathcal{A} - \mathcal{B}_i$ bimodules for $i = 1, 2$, and let $\mathcal{D}, \mathcal{E}$ be two type II$_1$ factors. Then*

(a) *If H_1 and H_2 satisfy the same conditions needed for Lemma 1.6, and if they are left-weakly equivalent, then $K_1 := L \otimes H_1$ is weakly equivalent to $K_2 := L \otimes H_2$ for any $\mathcal{D}$–$\mathcal{A}$ bimodule L.*

(b) *If H_1 and H_2 satisfy the same conditions needed for Lemma 1.6, and if they are right-weakly equivalent, then $K_1 := H_1 \otimes W$ is weakly equivalent to $K_2 := H_2 \otimes W$ for any $\mathcal{B}$–$\mathcal{E}$ bimodule W.*

Proof. (a) By Lemma 1.6, since H_1 is weakly equivalent to H_2 (and satisfies the conditions needed), there must exist a projection $p \in \mathcal{B}_1$ such that $H_1 p$ and X_2 are isomorphic as $\mathcal{A} - \mathcal{B}_2$ bimodules. This isomorphism extends in an obvious way to a spatial isomorphism between $L \otimes H_1 p = (L \otimes H_1)(1 \otimes p)$ and $L \otimes H_2$. Hence the claim follows from Lemma 1.6. The proof of item (b) follows like the one for item (a), using Remark 1.7. $\diamondsuit$

1.3. **Higher relative commutants.** Let $\mathcal{N} \subset \mathcal{M}$ be type II_1 factors with normalized trace tr. There exists a canonical extension $\mathcal{M}_1 \supset \mathcal{M}$, called Jones' basic construction for $\mathcal{N} \subset \mathcal{M}$, which is the von Neumann algebra generated by $\mathcal{M}$ acting via left multiplication on $L^2(\mathcal{M}, tr)$ and by the orthogonal projection $e_{\mathcal{N}}$ onto the subspace $L^2(\mathcal{N}, tr) \subset L^2(\mathcal{M}, tr)$. It is well-known that the Jones index $[\mathcal{M} : \mathcal{N}]$ is finite if and only if $\mathcal{M}_1$ is again a type II_1 factor; it is given by $[\mathcal{M} : \mathcal{N}] = 1/tr(e_{\mathcal{N}})$, with tr denoting the unique normalized trace on $\mathcal{M}_1$. We can apply the basic construction again for $\mathcal{M} \subset \mathcal{M}_1$ to obtain an extension $\mathcal{M}_2 \supset \mathcal{M}_1$. Iterating this construction, we obtain a sequence of II_1 factors $\mathcal{N} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ ...$. We obtain important invariants of the original inclusion $\mathcal{N} \subset \mathcal{M}$ via the so-called higher relative commutants $\mathcal{N}' \cap \mathcal{M}_k$ and $\mathcal{M}' \cap \mathcal{M}_k$. These are finite dimensional C*-algebras. If there exists a uniform bound for the dimensions of the centers of the relative commutants, the subfactor $\mathcal{N} \subset \mathcal{M}$ is called a *finite depth subfactor*. In this case, the inclusion diagram for $\mathcal{N}' \cap \mathcal{M}_{2k} \subset \mathcal{N}' \cap \mathcal{M}_{2k+1}$ does not depend on k for k sufficiently large; the corresponding graph is called the *principal graph* of $\mathcal{N} \subset \mathcal{M}$. Similarly, one defines the *dual principal graph* from the inclusion of $\mathcal{M}' \cap \mathcal{M}_{2k} \subset \mathcal{M}' \cap \mathcal{M}_{2k+1}$ for k sufficiently large. These graphs are important invariants for the inclusion $\mathcal{N} \subset \mathcal{M}$.

We have the following important results, which are presented in great detail and with precise references to original sources in [Bs]:

Proposition 1.9. *Let $\mathcal{N} \subset \mathcal{M}$ be a finite depth subfactor with finite index. Then*

(a) *The inclusions $\mathcal{N} \subset \mathcal{M}_{2k+1}$, $\mathcal{N} \subset \mathcal{M}_{2k}$, $\mathcal{M} \subset \mathcal{M}_{2k+1}$, $\mathcal{M} \subset \mathcal{M}_{2k}$ are given by the bimodule $\mathcal{M}^{\otimes k} = \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \otimes_{\mathcal{N}} \ ... \ \otimes_{\mathcal{N}} \mathcal{M}$ (k times), viewed, respectively, as an $\mathcal{N}$–$\mathcal{N}$, $\mathcal{N}$–$\mathcal{M}$, $\mathcal{M}$–$\mathcal{N}$ and $\mathcal{M}$–$\mathcal{M}$ bimodule.*

(b) *The embedding of $\mathcal{N}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_{k+1}$ coincides with the embedding of the algebras $\mathrm{End}_{\mathcal{M}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \subset \mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k})$ for k even. If k is odd, the embedding of $\mathcal{N}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_{k+1}$ coincides with the embedding of $\mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \subset \mathrm{End}_{\mathcal{M}-\mathcal{N}}(\mathcal{M}^{\otimes k+1})$, given by $x \in \mathrm{End}_{\mathcal{N}-\mathcal{N}}(\mathcal{M}^{\otimes k}) \to 1_{\mathcal{M}} \otimes x$.*

(c) *Analogous statements hold for the embedding of $\mathcal{M}' \cap \mathcal{M}_k \subset \mathcal{M}' \cap \mathcal{M}_{k+1}$; we only need to replace $\mathrm{Hom}_{\mathcal{X}-\mathcal{N}}$ by $\mathrm{Hom}_{\mathcal{X}-\mathcal{M}}$ in all the statements in (b), with $\mathcal{X} \in \{\mathcal{M}, \mathcal{N}\}$.*

Proof. Statement (a) is shown e.g. in [Bs], Proposition 3.2. Statement (b) can be found in [Bs], Corollary 4.2 and Corollary 4.4 (with tensoring from the right instead of tensoring from the left, as we have chosen here). Statement (c) follows from (b) and (a). $\diamondsuit$

Let $\mathcal{N}, \mathcal{M}, \mathcal{B}$ be type II_1 factors with $\mathcal{N} \subset \mathcal{M}$ a subfactor of finite index. Let $\{H_\lambda\}_\lambda$ and $\{K_\nu\}_\nu$ be a collection of mutually nonisomorphic irreducible $\mathcal{N}$–$\mathcal{B}$ and $\mathcal{M}$–$\mathcal{B}$ bimodules, respectively. Observe that $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$ is an $\mathcal{M}$–$\mathcal{B}$ bimodule for any $\mathcal{N}$–$\mathcal{B}$ bimodule H_λ. Similarly, we can view any $\mathcal{M}$–$\mathcal{B}$ bimodule K_ν as an $\mathcal{N}$–$\mathcal{B}$ bimodule by restricting the left action to $\mathcal{N}$. We say that the system of bimodules $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ is *closed under induction and restriction* if

- for each $\mathcal{N}$–$\mathcal{B}$ bimodule H_λ the $\mathcal{M}$–$\mathcal{B}$ bimodule $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$ is isomorphic to a direct sum of irreducible $\mathcal{M}$–$\mathcal{B}$ bimodules each of which is isomorphic to an element in $\{K_\nu\}_\nu$,
- for each $\mathcal{M}$–$\mathcal{B}$ bimodule K_ν the $\mathcal{N}$–$\mathcal{B}$ bimodule obtained from K_ν by restricting the left action to $\mathcal{N}$ is isomorphic to a direct sum of irreducible $\mathcal{N}$–$\mathcal{B}$ bimodules each of which is isomorphic to an element in $\{H_\lambda\}_\lambda$.

The *induction-restriction graph* for our system of bimodules is the bipartite graph whose (say) odd vertices are labeled by the elements in $\{H_\lambda\}_\lambda$ and whose even vertices are labeled by the elements in $\{K_\nu\}_\nu$. A vertex labeled by H_λ is connected with a vertex labeled by K_ν by L_λ^ν edges, where L_λ^ν is the multiplicity of H_λ in K_ν, viewed as an $\mathcal{N}$–$\mathcal{B}$ bimodule. By Frobenius reciprocity (see e.g. [Bs], Theorem 1.18), this number coincides with the multiplicity of K_ν in $\mathcal{M} \otimes_{\mathcal{N}} H_\lambda$.

Proposition 1.10. *Let $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ be a system of $\mathcal{N}$–$\mathcal{B}$- resp $\mathcal{M}$–$\mathcal{B}$-bimodules which is closed under induction and restriction.*

(a) *If $\{H_\lambda\}_\lambda$ contains a bimodule H_0 which is weakly isomorphic to the trivial $\mathcal{N}$–$\mathcal{N}$-bimodule $\mathcal{N}$, then the principal graph for $\mathcal{N} \subset \mathcal{M}$ is given by the connected component of the induction- restriction graph for $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ which contains H_0.*

(b) *If $\{K_\nu\}_\nu$ contains a bimodule K_0 which is weakly isomorphic to the trivial $\mathcal{M}$–$\mathcal{M}$-bimodule $\mathcal{M}$, then the dual principal graph for $\mathcal{N} \subset \mathcal{M}$ is given by the connected component of the induction- restriction graph for $(\{H_\lambda\}_\lambda, \{K_\nu\}_\nu)$ which contains K_0.*

Proof. Part (a) follows from Proposition 1.8 and Proposition 1.9,(b). Similarly, part (b) follows from Proposition 1.8 and Proposition 1.9,(c). $\diamondsuit$

Remark 1.11. In the setting of the last proposition, (a), there may be more than one bimodule H_λ which is weakly isomorphic to the trivial $\mathcal{N}$–$\mathcal{N}$-bimodule $\mathcal{N}$. The resulting graph will be independent of the choice of H_0.

Let H be an $\mathcal{A}$–$\mathcal{B}$ bimodule. We define $\mathrm{ind}(H)$ to be equal to the index $[\rho(\mathcal{B})' : \lambda(\mathcal{A})] = [\lambda(\mathcal{A})' : \rho(\mathcal{B})]$. In the following lemma, $(H_\lambda)_\lambda$ and $(K_\nu)_\nu$ are bimodules as in the last proposition, where we now assume for simplicity that they only denote the bimodules which label the vertices of a given principal graph. Moreover, we also assume the subfactor to be of finite depth, i.e., both sets only contain finitely many bimodules.

Lemma 1.12. *With notations as above, we have:*

(a) $\sum_\nu \mathrm{ind}(K_\nu) = \sum_\lambda \mathrm{ind}(H_\lambda)$.

(b) *Assume that the $\mathcal{A}$–$\mathcal{B}$-bimodule H decomposes as $H = \bigoplus m_i H_i$, with H_i irreducible $\mathcal{A}$–$\mathcal{B}$-bimodules, and let $l = \dim(\mathrm{End}_{\mathcal{A},\mathcal{B}}(H)) = \sum_i m_i^2$. Then we have $\mathrm{ind}(H_i) \geq \mathrm{ind}(H)/l$, with equality only if $\dim_\mathcal{A}(H_i) = m_i \dim_\mathcal{A}(H)/l$.*

Proof. It is well-known that the inclusion of higher relative commutants $\mathcal{M}' \cap \mathcal{M}_k \subset \mathcal{N}' \cap \mathcal{M}_k$ defines periodic commuting squares which generate in the limit a subfactor of index $[\mathcal{M} : \mathcal{N}]$. Hence we can use the results of [W1], Theorem 1.5,(iii). It follows that the index is equal to the quotient of the l^2-norms of the weight vectors of $\mathcal{M}' \cap \mathcal{M}_k$ and $\mathcal{N}' \cap \mathcal{M}_k$ for k sufficiently large. Let p_λ and p_μ be minimal idempotents in $\mathcal{M}' \cap \mathcal{M}_k$ and $\mathcal{N}' \cap \mathcal{M}_k$ respectively. Then we have $\mathrm{ind}(p_\nu \mathcal{M}_k) = tr(p_\nu)^2 [\mathcal{M} : \mathcal{N}]^k$ and $\mathrm{ind}(p_\lambda \mathcal{M}_k) = tr(p_\lambda)^2 [\mathcal{M} : \mathcal{N}]^{k+1}$. Solving for $tr(p_\lambda)^2$ and $tr(p_\nu)^2$, we obtain

$$[\mathcal{M} : \mathcal{N}] = \frac{\sum_\nu \mathrm{ind}(p_\nu \mathcal{M}_k)/[\mathcal{M} : \mathcal{N}]^k}{\sum_\lambda \mathrm{ind}(p_\lambda \mathcal{M}_k)/[\mathcal{M} : \mathcal{N}]^{k+1}}.$$

The claimed formula follows from this in the case that our system of bimodules labels the vertices of the principal graph. One obtains the claim for the dual principal graph by the same proof applied to the inclusion $\mathcal{M} \subset \mathcal{M}_1$.

Part (b) is proved using Lagrange multipliers as follows: Let $x_i = \dim_\mathcal{A}(H_i)$ and let $d = \dim_\mathcal{A} H$. Then the minimum of the function $f(x_1, \ldots x_r) = \sum x_i^2$ subject to the condition $\sum m_i x_i = d$ is obtained for $2x_i = \lambda m_i$, and we deduce from the constraint that $d = \frac{\lambda}{2} \sum m_i^2 = l\lambda/2$. Hence $x_i = m_i d/l$ and

$$\sum_i (\dim_\mathcal{A} H_i)^2 = \frac{d^2}{l^2} \sum_i m_i^2 = d^2/l. \qquad (*)$$

Now observe that if p_i is the projection onto the submodule $H_i \subset H$, we have $tr(p_i) = \dim_\mathcal{A}(H_i)/\dim_\mathcal{A}(H)$ and $\mathrm{ind}(H_i) = tr(p_i)^2 \mathrm{ind}(H)$ (again see [W1], Theorem 1.5,(iii)). The claim follows from this after multiplying $(*)$ by $\mathrm{ind}(H)/d^2$.

2. Categories

In this section we deal with categories which can be considered as generalizations of the representation categories of finite groups. This allows us to deal simultaneously with categories of bimodules of von Neumann factors, fusion categories (which can be constructed using quantum groups or loop groups) and categories obtained from unitary braid representations. For more details, we refer to [ML], [Fr] for general categorical notions, and to [Ks], [T] for tensor categories; our treatment of traces also uses results from [LR].

2.1. General definitions.

We recall some basic definitions and set up notations.

In the following, $\mathcal{C}$ will always denote a strict monoidal complex tensor category with unit $\mathbb{1}$. This means that $\mathcal{C}$ is a category with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product* which satisfies certain associativity conditions such as the *Pentagon Axiom.* There are similar axioms involving the morphisms $l_X : \mathbb{1} \otimes X \to X$ and $r_X : X \otimes \mathbb{1} \to X$ called the left and right *unit constraints*. Moreover, $\mathcal{C}$ being a complex category just means that the homomorphisms $\mathrm{Hom}(X, Y)$ form a complex vector space for any objects X and Y in $\mathcal{C}$.

The complex tensor category $\mathcal{C}$ is called a $*$ tensor category if there exists a contragredient complex conjugate functor $* : \mathcal{C} \to \mathcal{C}$ which is compatible with $\otimes$. This means in detail that:

- if $f \in \mathrm{Hom}(X, Y)$, then $f^* \in \mathrm{Hom}(Y, X)$,
- $(\alpha f)^* = \bar{\alpha} f^*$ for all $\alpha \in \mathbb{C}$ and $f \in \mathrm{Hom}(X, Y)$,
- $(fg)^* = g^* f^*$ for $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(U, X)$,
- $(f \otimes g)^* = f^* \otimes g^*$ for $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(U, V)$,
- $1_X^* = 1_X$ for the identity morphism 1_X for any object X in $\mathcal{C}$.

2.2. **Duality and Frobenius reciprocity.** An object X in a strict monoidal category $\mathcal{C}$ is called *left rigid* if there exists an object $\bar{X} \in \mathcal{C}$ and a pair of morphisms $i_X : \mathbb{1} \to X \otimes \bar{X}$ and $d_X : \bar{X} \otimes X \to \mathbb{1}$ such that the maps $(1_X \otimes d_X)(i_X \otimes 1_X) : X \to X$ and $(d_X \otimes 1_{\bar{X}})(1_X \otimes i_{\bar{X}}) : \bar{X} \to \bar{X}$ are 1_X and $1_{\bar{X}}$. An object X is called right rigid if we can find an object $\bar{X}'$ and morphisms $i'_X : \mathbb{1} \to \bar{X}' \otimes X$ and $d'_X : X \otimes \bar{X}' \to \mathbb{1}$ satisfying analogous identities. It is easy to check that in a $*$ category any left rigid object is also right rigid, with $\bar{X}' = \bar{X}$, $i'_X = d_X^*$ and $d'_X = i_X^*$. Hence we will in the following only talk about rigid objects. A category $\mathcal{C}$ is called rigid if every object of $\mathcal{C}$ is rigid.

With this notion of duality, we also have the usual Frobenius reciprocity isomorphism between $\mathrm{Hom}(V, W \otimes \bar{X})$ and $\mathrm{Hom}(V \otimes X, W)$ for any objects V, W in $\mathcal{C}$. One checks easily that these isomorphisms are given by the maps $a \to (1_W \otimes d_X) \circ (a \otimes 1_X)$ and $b \to (b \otimes 1_Y) \circ (1_V \otimes i_X)$ for $a \in \mathrm{Hom}(V, W \otimes X)$ and $b \in \mathrm{Hom}(V \otimes Y, W)$. In particular, one obtains as a special case that $\dim \mathrm{Hom}(\mathbb{1}, X \otimes \bar{X}) = \dim \mathrm{End}(X) = 1$ if X is a simple object. Hence the morphisms i_X and d_X are unique up to scalar multiples for X simple. We shall say that the rigidity morphisms i_X and d_X are *normalized* if $i_X^* i_X = d_X d_X^* *$.

2.3. **Dimension, trace and conditional expectation.** In the following we always assume the rigidity morphisms i_X and d_X to be normalized for any object X. If X is simple, this can always be assumed after some rescaling in view of the discussion in the last section. For normalized rigidity morphisms, we can now define the dimension of a simple object X to be equal to the scalar

$$\dim(X) = i_X^* i_X = d_X d_X^*.$$

Of course, we would like the dimension to be additive with respect to a decomposition $X = \oplus W_i$, with the W_i being simple objects. To do so, we define morphisms $\phi_i : W_i \to X$ such that $\phi_i^* \phi_j = \delta_{ij} 1_{W_i}$ and $\sum_i \phi_i \phi_i^* = 1_X$, and we define

$$i_X = \sum (\phi_i \otimes \bar{\phi}_i) i_{W_i}, \quad d_{X \otimes Y} = \sum d_{W_i} (\bar{\phi}_i^* \otimes \phi_i^*), \tag{2.1}$$

where the $\bar{\phi}_i$ are the analogous morphisms for the decomposition of the dual $\bar{X} = \sum \oplus_i \bar{W}_i$. Then it is easy to check that these morphisms satisfy the rigidity axiom, and they are normalized if the ϕ_i are so. Moreover, one also checks that these morphisms yield the desired additivity property of the dimension function.

Additionally, the dimension function should be multiplicative with respect to the tensor product. If $X \otimes Y$ is a tensor product of simple objects X and Y, we obtain normalized rigidity morphisms

$$i_{X\otimes Y} = (1_X \otimes i_Y \otimes 1_{\bar{X}})i_X, \quad d_{X\otimes Y} = d_Y(1_{\bar{Y}} \otimes d_X \otimes 1_X).$$

It can be shown that these rigidity morphisms define the same dimension as the one we obtain from the decomposition $X \otimes Y \cong \oplus_i W_i$, with W_i simple and with rigidity morphisms as defined in the last paragraph. It will be convenient to represent the rigidity morphisms i_X and d_X, by the following pictures:

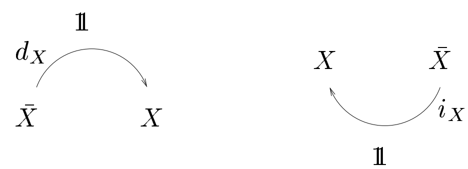

Figure 2.1. Rigidity morphisms

In a $*$ tensor category we define the *categorical trace* of an endomorphism $f \in \mathrm{End}(X)$ by

$$\mathrm{Tr}_X(f) = i_X^* \circ (f \otimes 1_{\bar{X}}) \circ i_X \in \mathrm{End}(\mathbb{1}). \tag{2.2}$$

If $Z = \oplus m_i X_i$, where X_i is a simple object, and m_i is the multiplicity of X_i in Z, we can write an element $f \in \mathrm{End}(Z)$ in the form $f = \oplus f_i$, where $f_i \in \mathrm{End}(m_i X_i)$ can be viewed as an $m_i \times m_i$ matrix. Defining rigidity morphisms i_Z, d_Z with respect to this decomposition, and using Equation 2.1, one checks easily that

$$Tr_Z(f) = \sum \dim(X_i) Tr(f_i),$$

where $Tr(f_i)$ is the usual trace of a matrix. This shows that we obtain a well-defined trace for $\mathrm{End}(Z)$ for any object Z, and that $Tr_Z(fg) = Tr_Z(gf)$ for any $f, g \in \mathrm{End}(Z)$. Moreover, using this formula, one shows as well that we can define the trace also by

$$\mathrm{Tr}_X(f) = i_{\bar{X}}^* \circ (1_{\bar{X}} \otimes f) \circ i_{\bar{X}} \in \mathrm{End}(\mathbb{1}).$$

This shows that $*$-categories satisfy the axioms of a spherical category (see [BWs]).

The *normalized trace* tr_X on $\mathrm{End}(X)$ is defined by $\mathrm{tr}_X(f) = \mathrm{Tr}_X(f)/(\dim X)$. In the following we will often just write Tr, tr for the trace or normalized trace when it is clear for which object it is defined.

Conditional expectations can also be very naturally defined using our categorical definitions. Let X be an object. Let $A = \mathrm{End}(X) \cong A \otimes 1_V \subset B = \mathrm{End}(X \otimes V)$. We define the map ε_A from B onto A by

$$\varepsilon_A(b) = \frac{1}{\dim V}(1_X \otimes i_V^*)(b \otimes 1_{\bar{V}})(1_X \otimes i_V);$$

in the tangle picture, $\varepsilon_A(b)$ is obtained from b by closing up the tangle with color V and renormalizing by $1/\dim V$.

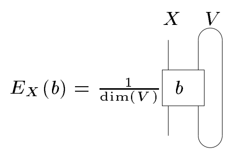

FIGURE 2.2. Conditional expectation

It is known and easy to check that this definition of conditional expectation coincides with the usual definition of conditional expectation in operator algebras (see e.g. [OW], Proposition 1.4). Actually, one can show more: Let X_1, X_2, X_3 be objects in our $*$ tensor categoy $\mathcal{C}$. Define the algebras $A = \mathrm{End}(X_2)$, $B = \mathrm{End}(X_1 \otimes X_2)$, $C = \mathrm{End}(X_2 \otimes X_3)$ and $D = \mathrm{End}(X_1 \otimes X_2 \otimes X_3)$. We can consider all these algebras as subalgebras of D, e.g. by identifying A with $1_{X_1} \otimes \mathrm{End}(X_2) \otimes 1_{X_3}$.

Proposition 2.1. *The algebras A, B, C, D form a commuting square, i.e. we have* $\varepsilon_B \varepsilon_C = \varepsilon_A = \varepsilon_C \varepsilon_B$.

2.4. **Braided tensor categories.** A strict monoidal category $\mathcal{C}$ is called *braided* if there exists a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ called the *braiding* such that:

$$c_{X,Y\otimes Z} = (1_Y \otimes c_{X,Z})(c_{X,Y} \otimes 1_Z)$$

and

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes 1_Y)(1_X \otimes c_{Y,Z}).$$

Naturality means that for any morphisms $f : X \to X'$ and $g : Y \to Y'$

$$(g \otimes f) \circ c_{X,Y} = c_{X',Y'} \circ (f \otimes g).$$

Finally, we also require that $c_{\mathbf{1},X} = 1_X = c_{X,\mathbf{1}}$ under the isomorphisms $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$.

2.5. C^* **tensor categories.** We call a complex $*$ tensor category a C^* tensor category if

(a) for any objects X, Y in $\mathcal{C}$ the space $\mathrm{Hom}(X, Y)$ is a Hilbert space with inner product $(a, b) = Tr(b^*a)$ for $a, b \in \mathrm{Hom}(X, Y)$,
(b) for any object X, Y in $\mathcal{C}$ the algebra $\mathrm{End}(Y)$ is a C^*-algebra acting on the Hilbert space $\mathrm{Hom}(X, Y)$.

Observe that these definitions imply that the dimensions of all objects are positive. A *braided C^* tensor category* is a C^* tensor category with a braiding for which all its braiding morphisms are unitary operators. For examples of C^*-tensor categories, see Section 6.1.

3. The multisided construction

3.1. **Categorical setting.** We shall use the following conventions: Let $\mathcal{C}$ be a finite braided C^* tensor category, where finite means that we only have finitely many equivalence classes of simple objects in $\mathcal{C}$. Let $\{X_\lambda, \lambda \in \Lambda\}$ be a set of representative nonequivalent simple objects, indexed by some labeling set Λ. We define d_λ to be the dimension of X_λ. We shall also assume that the category $\mathcal{C}$ is generated by an object X, i.e. that any simple object appears in some tensor power of X. We define $k = k(X) = gcd\{n, \mathbb{1} \subset X^{\otimes n}\}$. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ generated by the simple objects in $X^{\otimes mk}$, $m \in \mathbb{N}$. We define algebras $A_n = \mathrm{End}(X^{\otimes n}) = \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$. By definition of A_n, the simple components of A_n are labeled by the equivalence classes of simple objects which appear in the n-th tensor power of X, i.e. by a certain subset Λ_n of Λ. We define the embeddings $\iota_r : a \in A_n \to a \otimes 1_r \in A_{n+r}$, where we will often omit the subscript r. It follows from the definitions that the vertices of the inclusion diagram for $\iota : A_n \to A_{n+1}$ are labeled by the elements of Λ_n and Λ_{n+1} respectively; the vertex labeled by $\lambda \in \Lambda_n$ is connected with the one labeled by μ by L_λ^μ edges, where L_λ^μ is the multiplicity of the object X_μ in $X_\lambda \otimes X$. We have the following commuting diagram of embeddings

$$\begin{array}{ccccc} & 1_m \otimes A_n & \subset & A_{n+m} & \\ 1 \otimes \iota & \downarrow & & \downarrow & \iota \\ & 1_m \otimes A_{n+1} & \subset & A_{n+m+1} & \end{array} \tag{3.1}$$

We have the following simple

Lemma 3.1. *Let $\mathcal{C}$ be a finite C^*-tensor category, not necessarily braided. Then we have*

(a) *$\Lambda_n = \Lambda_{n+k}$ for n sufficiently large; in particular $\Lambda' := \Lambda_{nk}$ for n sufficiently large labels the simple objects of $\mathcal{C}'$.*
(b) *The weight vector for the trace on the algebra A_n is $\vec{v}_n = (d_\lambda/(\dim X)^n)_{\lambda\in\Lambda_n}$.*
(c) *The inductive limit of $(1_m \otimes A_n \subset A_{n+m})$ defines an inclusion $B \subset A$ of hyperfinite II_1 factors with index $(\dim X)^{2m}$.*
(d) *$\sum_{\lambda\in\Lambda_n} d_\lambda^2 = \frac{1}{k}\sum_{\lambda\in\Lambda} d_\lambda^2$ for n sufficiently large.*

Proof. If the trivial object $\mathbb{1}$ appears in the r-th tensor power of X and $X_\lambda \subset X^{\otimes n}$, then we have

$$X_\lambda \cong X_\lambda \otimes \mathbb{1} \subset X_\lambda \otimes X^{\otimes r} \subset X^{\otimes n+r}.$$

Hence $\Lambda_n \subset \Lambda_{n+r}$ for all $n \in \mathbb{N}$. As Λ is finite, these inclusions become equalities for n sufficiently large. Applying this to any r such that $\mathbb{1} \subset X^{\otimes r}$, we can similarly prove $\Lambda_n = \Lambda_{n+k}$ for k the *gcd* of all such r and n sufficiently large. This shows (a).

Statement (b) follows from the fact that the value of the normalized trace of a projection p_λ corresponding to a simple object $X_\lambda \subset X^{\otimes n}$ is given by $tr(p_\lambda) = d_\lambda/(\dim X)^n$.

For statement (c) observe that Diagram 3.1 defines a commuting square by Proposition 2.1. Moreover, the sequence of algebras as in the statement has a k-periodic inclusion pattern: by part (a), we have the same labeling sets for the algebras in Diagram 3.1 if we substitute n by $n+k$ everywhere, for n sufficiently large. Moreover, also the inclusion pattern remains the same by the discussion before Diagram 3.1. It follows from [W1], Theorem 1.5,(iii), that the index $[A:B]$ is given by the ratio $\|\vec{v}_n\|^2/\|\vec{v}_{n+1}\|^2$, for n large enough. As this holds for any sufficiently large n, we have

$$[A:B]^k = \prod_{i=1}^{k} \frac{\|\vec{v}_{n-1+i}\|^2}{\|\vec{v}_{n+i}\|^2} = \frac{\|\vec{v}_n\|^2}{\|\vec{v}_{n+k}\|^2}.$$

The claim now follows from the fact that $\vec{v}_n = (\dim X)^k \vec{v}_{n+k}$, by (a) and (b). Finally observe that $(\dim X)^2\|\vec{v}_{n+1}\|^2 = \|\vec{v}_n\|^2$ implies $\sum_{\lambda\in\Lambda_n}(d_\lambda)^2 = \sum_{\mu\in\Lambda_{n+1}}(d_\mu)^2$ for all n sufficiently large. As $\Lambda_n \cap \Lambda_m = \emptyset$ whenever $|n-m| < k$, we obtain Statement (d). $\diamondsuit$

3.2. **Multisided Construction.** The subfactors constructed in the last section will sometimes be denoted as one-sided subfactors. We will now generalize the construction in [E1] to the setting of braided C^*-tensor categories, which we call multisided subfactors in analogy to the notation in [E1]. We will fix a positive integer s. For the s-sided construction, we will have to define an embedding of algebras $A_n^{\otimes s} \subset A_{ns}$ such that we will obtain a subfactor if we consider the inductive limit over n.

We shall need special braids $\gamma_n \in \mathcal{B}_{sn}$, which can be defined inductively by $\gamma_1 = 1_s$ and by Figure 3.3. Alternatively, the braid γ_n can be described as follows: arrange the

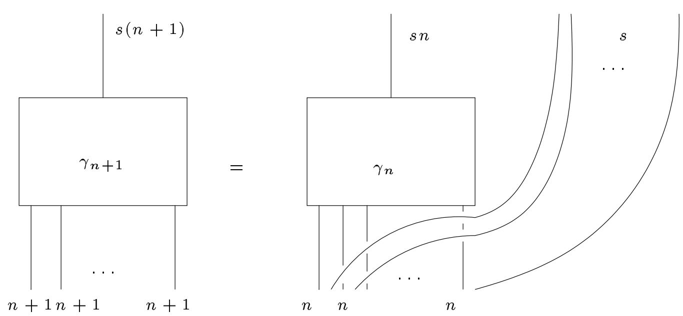

FIGURE 3.3. Inductive property of intertwining braids.

points labeled by the numbers 1 up to ns in a rectangular pattern with height n and width s. Now we can numerate the points either by first going down the columns, or by first going to the right in each row. This defines a permutation π mapping the i-th point in the column-first count to the i-th point in the row-first count. The braid γ_n is now defined by this permutation where the i-th lower point is connected with the $\pi(i)$-th upper point and where we assume all crossings to be positive (i.e., the strand going from southwest to northeast crosses over the one going from southeast to northwest). A picture for this braid can be found in [E2], page 83.

Let $c = c_{X,X}$ be the braiding morphism for X. By definition, we obtain a unitary representation ρ of the braid group $\mathcal{B}_n$ into A_n by mapping the generator σ_i to $c_i = 1_{i-1}\otimes c\otimes 1_{n-1-i}$. We define the unitary $u_n = u_n^{(s)} = \rho(\gamma_n)$, with γ_n defined as in Figure 3.3. Finally, the embedding from $A_n^{\otimes s}$ into A_{ns} is given by first identifying $A_n^{\otimes s}$ with $\mathrm{End}(X^{\otimes n})^{\otimes s} \subset \mathrm{End}(X^{\otimes ns}) = A_{ns}$ and by then conjugating this with u_n, i.e. by

$$(a_1 \otimes \cdots \otimes a_s) \xmapsto{\hat{u}_n} u_n(a_1 \otimes \cdots \otimes a_s)u_n^*;$$

throughout this paper, $\hat{u}$ will denote the inner automorphism given by conjugation via the unitary u unless stated otherwise. We now obtain the following diagram of maps, where the vertical arrows are labeled by $\iota^{\otimes s} = \iota_1^{\otimes s}$ and $\iota = \iota_s$ respectively:

$$\begin{array}{ccc} & \hat{u}_n & \\ A_n^{\otimes s} & \longrightarrow & A_{ns} \\ \downarrow & & \downarrow \\ A_{n+1}^{\otimes s} & \longrightarrow & A_{(n+1)s} \\ & \hat{u}_{n+1} & \end{array} \tag{3.2}$$

Then we have the following lemma which has essentially already been proved in [E1], Section 3.2; the case proved there would correspond to the special case in which A_n is generated by the image of $\mathcal{B}_n$.

Lemma 3.2. *The diagram 3.2 above commutes and also forms a commuting square. Moreover, the inclusion pattern is k-periodic.*

Proof. We check first that Diagram 3.2 is a commuting diagram: This is most easily seen by the following pictures (these proofs by pictures contain all the necessary details and translate faithfully to the algebraic proofs by simply re-writing the definitions already included in this article). We take $s = 3$ for simplicity. For $b \in \mathcal{A}_n^{\otimes s}$, we have

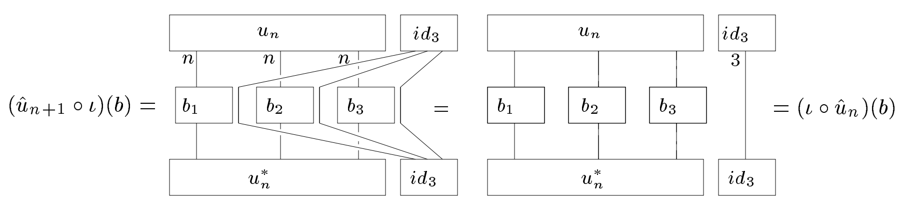

FIGURE 3.4. Diagram 3.2 is a commuting diagram.

Now we check that Diagram 3.2 is a commuting square, i.e., that $(E_{A_{sn}} \circ \hat{u}_{n+1})(b) = (\hat{u}_n \circ E_{A_n^{\otimes s}})(b)$ for $b \in A_{n+1}^{\otimes s}$. We use the categorical definition for a conditional expectation as described in Subsection 2.3, Figure 2.2. For $b = b_1 \otimes \cdots \otimes b_s \in A_{n+1}^{\otimes s}$, we have

$(E_{A_{sn}} \circ \hat{u}_{n+1})(b) =$

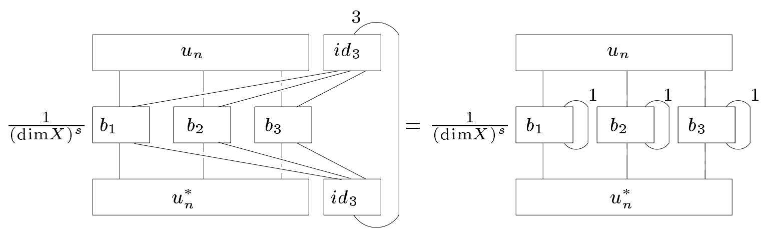

FIGURE 3.5. Diagram 3.2 is a commuting square.

$= (\hat{u}_n \circ E_{A_n^{\otimes s}})(b)$. To show that the inclusion diagrams are k-periodic for large n, observe that Lemma 3.1(a) implies that we have a 1-1 correspondence between the labeling sets of simple components of $A_n^{\otimes s}$ and $A_{n+k}^{\otimes s}$ as well as between the components of A_{ns} and $A_{(n+1)s}$. This identification of edges is compatible with the number of edges between them, which again is just given by tensor product multiplicites. $\diamondsuit$

Theorem 3.3. *Fix $s \in \mathbb{N}$, $s > 1$. Then there exists a subfactor $\mathcal{N} \subset \mathcal{M}$ with the embedding $\mathcal{N} \hookrightarrow \mathcal{M}$ given by $\hat{u} := \varinjlim \hat{u}_n : \varinjlim A_n^{\otimes s} \to \varinjlim A_{ns}$. Its index is equal to $(\sum_{\lambda \in \Lambda'} d_\lambda^2)^{s-1}$, where Λ' is an indexing set for the simple objects of the subcategory $\mathcal{C}'$ as defined at the beginning of this subsection and $d_\lambda = \dim(X_\lambda)$.*

Proof. This was done in [E1] in the case that the A_n's are generated by only braid elements. By Lemma 3.2, Diagram 3.2 is a periodic commuting square for large n. Thus, by [W1], Theorem 1.5,(iii), $\hat{u} : \mathcal{N} \hookrightarrow \mathcal{M}$ is an inclusion of hyperfinite II_1 factors with index given by $\|\vec{t}_n\|^2/\|\vec{v}_n\|^2$ for n sufficiently large, where $\vec{t}_n$ and $\vec{v}_n$ are the trace vectors for the trace in $\mathcal{M}$ restricted to the finite dimensional approximants $A_n^{\otimes s}$ and A_{ns}, respectively. For this observe that if $k|n$ the dimension vectors for $A_n^{\otimes s}$ and A_{ns} are given given by $\vec{t}_{ns} = (d_{\vec{\lambda}}/(\dim X)^{ns})_{\vec{\lambda}}$ and $\vec{v}_{ns} = (d_\nu/(\dim X)^{ns})_\nu$, with $\vec{\lambda} \in (\Lambda')^s$ and $\nu \in \Lambda'$; here $d_{\vec{\lambda}} = \prod_{i=1}^s d_{\lambda_i}$. Hence we obtain

$$[\mathcal{M} : \mathcal{N}] = \frac{\|\vec{t}_n\|^2}{\|\vec{v}_n\|^2} = \frac{\sum_{\vec{\lambda} \in (\Lambda')^s} d_{\vec{\lambda}}^2}{\sum_{\nu \in \Lambda'} d_\nu^2} = (\sum_{\lambda \in \Lambda'} d_\lambda^2)^{s-1} \quad \diamondsuit.$$

3.3. **More embeddings.** We shall need a variation of the embeddings in the last section for the construction of certain bimodules.

Lemma 3.4. *Let $\vec{m} = (m_1, \cdots, m_s)$, where $m_i \in \mathbb{Z}_{\geq 0}$, and $m_1 \geq m_2 \geq \cdots \geq m_s$. Then there exist unitaries $u_{\vec{m},n} = u_{\vec{m},n}(s) \in A_{|\vec{m}|+sn}$ such that we obtain k periodic commuting* *squares*

$$
\begin{array}{ccc}
A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} & \overset{\hat{u}_{\vec{m},n}}{\longrightarrow} & A_{|\vec{m}|+ns} \\
\downarrow & & \downarrow \\
A_{n+1+m_1} \otimes \cdots \otimes A_{n+1+m_s} & \underset{\hat{u}_{\vec{m},n+1}}{\longrightarrow} & A_{|\vec{m}|+(n+1)s}
\end{array}
\tag{3.3}
$$

which produce an inclusion of hyperfinite II_1 factors which is isomorphic to the one in Theorem 3.3. It will also be denoted by $\mathcal{N} \subset \mathcal{M}$.

Proof. The unitaries $u_{\vec{m},n} = u_{\vec{m},n}(s) \in A_{|\vec{m}|+sn}$ are defined from the unitaries from before, $u_n(l)$, $(l = 1, \cdots, s)$. We shall give diagramatic representations of these unitaries. Let $t_{\vec{m}} = t_{\vec{m}}(s)$ be the unitary in $A_{\vec{m}}$ given by the picture in Figure 3.6, where the unitary $u_r^{(s)}$ is given by Figure 3.3 for $s > 1$, and it is equal to id_r for $s = 1$, and any positive integer r.

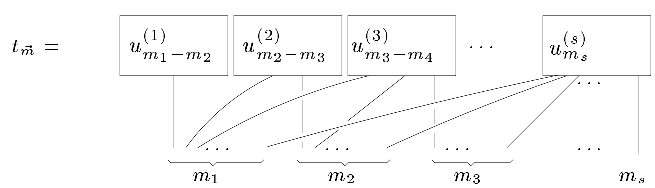

Figure 3.6

Then, the unitary $u_{\vec{m},n}$ will be defined from $t_{\vec{m}}$ and $u_n^{(s)}$ in Figure 3.7.

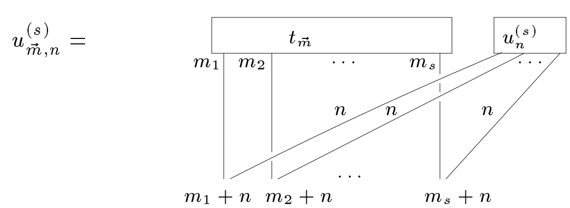

Figure 3.7

We proceed as in Lemma 3.2 to show that Diagram 3.3 is a commuting square. First we check that our diagram is a commuting diagram; we shall denote the vertical arrows by $\iota^{\otimes s}$ and ι respectively. Assume $s = 3$ again for simplicity.

For $b \in A_{n+m_1} \otimes \cdots \otimes A_{n+m_s}$, we have $(\hat{u}_{\vec{m},n+1} \circ \iota^{\otimes s})(b) =$

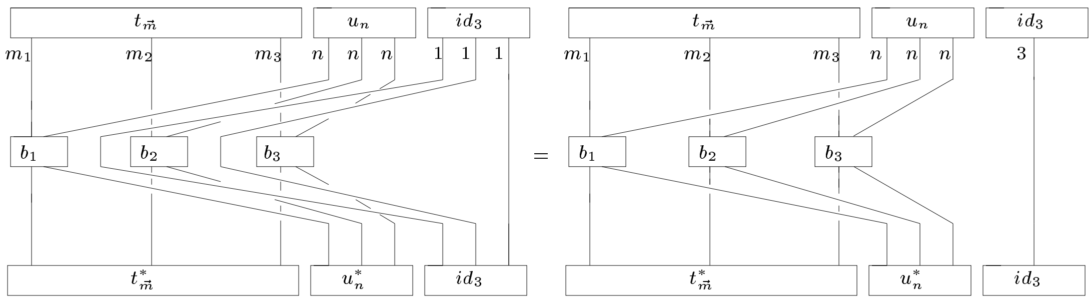

FIGURE 3.8. Diagram 3.3 is a commuting diagram.

$= (\iota \circ \hat{u}_{\vec{m},n})(b)$. The commuting square property as well as k periodicity is shown in the same way as in Lemma 3.2.

It remains to show that the subfactor constructed in this lemma is conjugate to the one in Theorem 3.3. We define an automorphism Φ of the factor $\mathcal{M} = \varinjlim A_{sn+|\vec{m}|} = \varinjlim A_{s(n+m_1)}$ that will carry the subfactor $\hat{u}(\mathcal{N}) = \varinjlim u_n A_n^{\otimes s} u_n^*$ to the subfactor defined here, $\hat{u}_{\vec{m}}(\mathcal{N}) = \varinjlim u_{\vec{m},n}(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s})u^*_{\vec{m},n}$. Define Φ_n at the finite dimensional level by

$$a \in A_{sn+|\vec{m}|} \mapsto u_{n+m_1} b_n \iota(u^*_{\vec{m},n} a u_{\vec{m},n}) b_n^* u^*_{n+m_1} \in A_{s(n+m_1)},$$

where $\iota : A_{sn+|\vec{m}|} \to A_{s(n+m_1)}$ is the usual inclusion, and where $b_n \in A_{s(n+m_1)}$ is a unitary described by the picture below. Observe that $b_n \iota(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \otimes 1_{sm_1 - |\vec{m}|}) b_n^*$ equals the image of the natural inclusion map $A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \to A^{\otimes s}_{n+m_1}$ (recall $m_1 \geq m_i$).

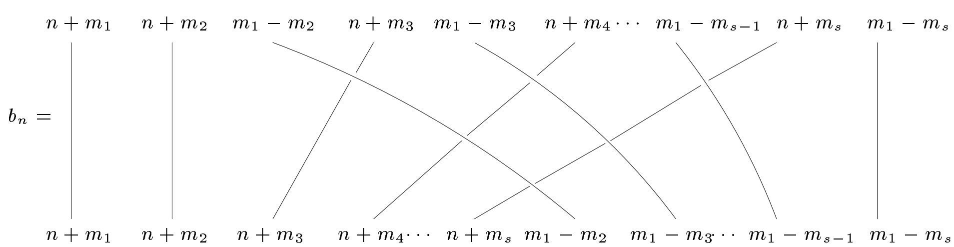

FIGURE 3.9. Pictorial description of $b_n \in A_{s(n+m_1)}$.

It is easy to check that the maps Φ_n are compatible with the inclusions for n to $n+1$, and so we can define $\Phi = \varinjlim \Phi_n$. We observe that $u_{\vec{m},n}(a_1 \otimes \cdots \otimes a_s)u^*_{\vec{m},n} \overset{\Phi}{\mapsto} u_{n+m_1}(a_1 \otimes \cdots \otimes a_s)u^*_{n+m_1}$ for $a_i \in A_{n+m_i}$, so that Φ carries $\hat{u}(\mathcal{N})$ to $\hat{u}_{\vec{m}}(\mathcal{N})$. It is easy and also left to the reader to check that Φ is an automorphism. $\diamondsuit$

3.4. **Endomorphisms.** We now want to construct bimodules with respect to the just constructed factors $\mathcal{N}$ and $\mathcal{M}$ in the proof of the last theorem. This will be done according to the recipe described in Remark 1.3. To do so, we need to define the endomorphisms mentioned in the braid setting before, in the categorical setting.

Lemma 3.5. *Fix $m_i \in \mathbb{Z}_{\geq 0}$, $i = 1, 2, \ldots, s$, with $m_1 \geq m_2 \geq \cdots \geq m_s$.*

(a) *For $n \in \mathbb{N}$, the maps*

$$\begin{array}{rcl} A_n^{\otimes s} & \longrightarrow & A_{m_1+n} \otimes \cdots \otimes A_{m_s+n} \\ a_1 \otimes \cdots \otimes a_s & \mapsto & (1_{m_1} \otimes a_1) \otimes \cdots \otimes (1_{m_s} \otimes a_s) \end{array}$$

extend to an endomorphism $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$, where $\vec{m} := (m_1, \ldots, m_s)$.

(b) *Let $\hat{u}$ denote the embedding of $\mathcal{N} \hookrightarrow \mathcal{M}$. The endomorphism $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}$ extends to an endomorphism of $\mathcal{M}$, denoted by $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$, that is, the following is a commuting diagram:*

$$\begin{array}{ccccc} & & \hat{u} & & \\ & \mathcal{N} & \hookrightarrow & \mathcal{M} & \\ \mathrm{Shift}_{\vec{m}}^{\mathcal{N}} & \downarrow & & \downarrow & \mathrm{Shift}_{\vec{m}}^{\mathcal{M}} \\ & \mathcal{N} & \hookrightarrow & \mathcal{M} & \\ & & \hat{u}_{\vec{m}} & & \end{array}$$

(c) $(\mathrm{Shift}_{\vec{m}}^{\mathcal{M}} \circ \hat{u})$ *only depends on the norm $|\vec{m}|$ of $\vec{m}$, and it is of the form*

$$\begin{array}{rcl} A_n^{\otimes s} & \longrightarrow & A_{|\vec{m}|+sn} \\ (a_1 \otimes \cdots \otimes a_s) & \mapsto & 1_{|\vec{m}|} \otimes u_n (a_1 \otimes \cdots \otimes a_s) u_n^*. \end{array}$$

Proof. (a) Let $v_{\vec{m},n} \in A_{|\vec{m}|+sn}$ be the unitary image under ρ of the braid described by Figure 3.10. Then it is easy to see pictorially that for any element $a_1 \otimes \cdots \otimes a_s \in A_n^{\otimes s}$, the maps defined in the statement of (a) are given by

$$(a_1 \otimes \cdots \otimes a_n) \;\mapsto\; v_{\vec{m},n} (a_1 \otimes \cdots \otimes a_n \otimes \mathrm{id}_{|\vec{m}|}) v_{\vec{m},n}^* \quad \in A_{n+m_1} \otimes \cdots \otimes A_{n+m_s}.$$

$v_{\vec{m},n} =$

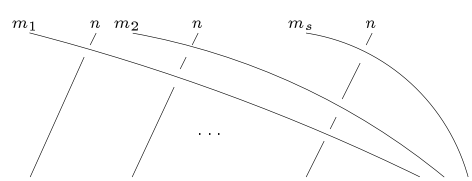

FIGURE 3.10

The fact that these maps extend to the von Neumann algebra inductive limit $\mathcal{N} = \varinjlim A_n^{\otimes s}$ follows from the fact that the following are commuting diagrams with respect to the canonical inclusions:

$$\begin{array}{ccccc} A_n^{\otimes s} & \hookrightarrow & A_n^{\otimes s} \otimes A_{|\vec{m}|} & \overset{\hat{v}_{\vec{m},n}}{\longrightarrow} & A_{n+m_1} \otimes \cdots \otimes A_{n+m_s} \\ \downarrow & & \downarrow & & \downarrow \\ A_{n+1}^{\otimes s} & \hookrightarrow & A_{n+1}^{\otimes s} \otimes A_{|\vec{m}|} & \underset{\hat{v}_{\vec{m},n+1}}{\longrightarrow} & A_{n+1+m_1} \otimes \cdots \otimes A_{n+1+m_s} \end{array} \tag{3.4}$$

and from the fact that the maps are norm and trace preserving. We denote the resulting endomorphism by $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}$.

(b) We shall extend the map $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}$ to $\mathcal{M}$ after embedding $\mathcal{N}$ in $\mathcal{M}$ via $\hat{u}$ (given by the inductive limit of conjugation of unitaries u_n or $u_{\vec{m},n}$ as in Figures 3.7 and 3.3). At the finite dimensional level we define $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}} : \mathcal{M} = \varinjlim A_{sn} \longrightarrow \mathcal{M} = \varinjlim A_{|\vec{m}|+sn}$ as follows:

$$\widehat{\omega_n} : \quad A_{sn} \hookrightarrow A_{|\vec{m}|+sn} \longrightarrow A_{|\vec{m}|+sn}, \tag{3.5}$$

where the first arrow stands for the standard inclusion $a \in A_{sn} \mapsto a \otimes 1 \in A_{|\vec{m}|+sn}$, and where the second arrow stands for conjugation by the unitary $\omega_n = \omega_n(s, \vec{m}) \in A_{|\vec{m}|+sn}$ defined by

$$\omega_n := u_{\vec{m},n} v_{\vec{m},n} (u_n^* \otimes \mathrm{id}_{|\vec{m}|}); \tag{3.6}$$

here $u_{\vec{m},n}$ and $v_{\vec{m},n}$ are given by Figures 3.6, 3.7, and 3.10. We give a diagramatic representation for $s = 3$ in Figure 3.11, with $b \in A_{sn}$:

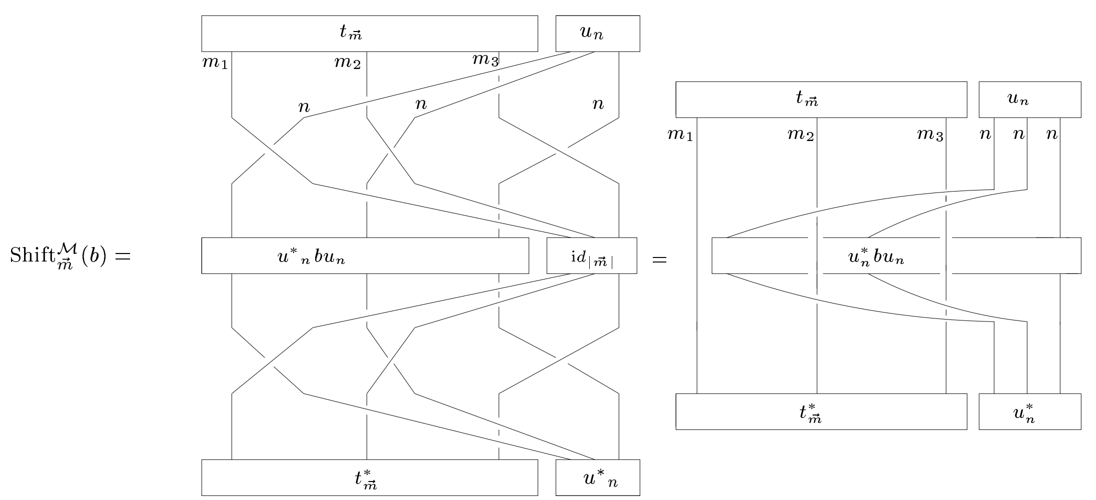

FIGURE 3.11. Pictorial representation of $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(b) \in A_{|\vec{m}|+sn}$, for $b \in A_{sn}$ (s=3).

We want to show that these maps extend to a well-defined map $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$ on the inductive limit $\varinjlim A_{sn}$, i.e., we have to show that $\widehat{\omega}_{n+1}(\iota(b)) = \iota\big(\widehat{\omega}_n(b)\big)$, where we use the notation ι for the standard inclusions of $A_{sn} \to A_{s(n+1)}$ as well as for $A_{|\vec{m}|+sn} \to A_{|\vec{m}|+s(n+1)}$. To show this, we need the inductive property of the unitaries $u_n^{(s)}$ mentioned already at the braid level, seen in Figure 3.3, to write $u_{\vec{m},n+1}$ in terms of $u_{\vec{m},n}$ and of id_s. We then have for $b \in A_{sn}$ that $\widehat{\omega}_{n+1}(\iota(b)) =$

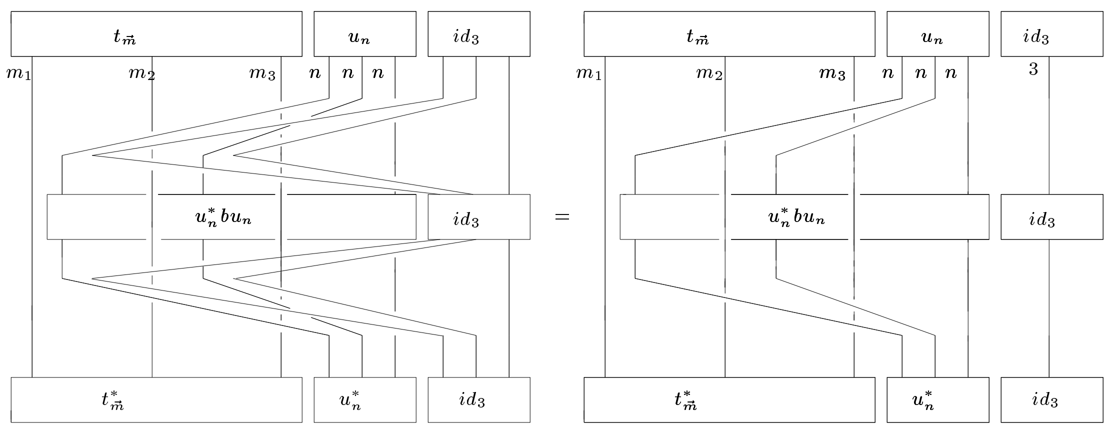

FIGURE 3.12. $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$ is well-defined.

$= \iota\big(\widehat{\omega}_n(b)\big)$. Hence $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}} = \varinjlim \widehat{\omega}_n$ is well defined.

We still need to show that $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$ 'extends' $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}$, i.e., that $(\mathrm{Shift}_{\vec{m}}^{\mathcal{M}} \circ \hat{u}) = (\hat{u} \circ \mathrm{Shift}_{\vec{m}}^{\mathcal{N}})$. From definition, for $a = a_1 \otimes \cdots \otimes a_s \in A_n^{\otimes s}$,

$$\begin{aligned}
(\mathrm{Shift}_{\vec{m}}^{\mathcal{M}} \circ \lim_{\rightarrow} \hat{u}_n)(a) &= (\hat{\omega}_n \circ \iota \circ \hat{u}_n)(a) \\
&= (\hat{u}_{\vec{m},n} \circ \hat{v}_{\vec{m},n})(a \otimes \mathrm{id}_{|\vec{m}|}) \\
&= (\lim_{\rightarrow} \hat{u}_{\vec{m},n} \circ \mathrm{Shift}_{\vec{m}}^{\mathcal{N}})(a).
\end{aligned}$$

Because of this, we shall after this lemma drop the superscripts and write $\mathrm{Shift}_{\vec{m}}$ for either $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$ or $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}}$.

(c) This follows from the definition. Take $(a_1 \otimes \cdots \otimes a_s) \in A_n^{\otimes s}$.

Using Figure 3.11, we obtain that $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(u_n(a_1 \otimes a_2 \otimes a_3)u_n^*) =$

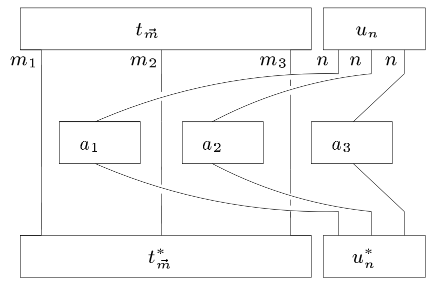

FIGURE 3.13

$= 1_{|\vec{m}|} \otimes u_n(a_1 \otimes a_2 \otimes a_3)u_n^*$. ♢

Proposition 3.6. *Let* $\mathrm{Shift}_{\vec{m}}$ *be as in Lemma 3.5.*

(a) $\mathrm{Shift}_{\vec{m}}(\mathcal{M}) \subset \mathcal{M}$ *is an inclusion of II_1 factors with index* $(\dim(X))^{2|\vec{m}|}$, *where* $|\vec{m}| = \sum m_i$ *and* $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ *has a subalgebra isomorphic to* $A_{m_1} \otimes \cdots \otimes A_{m_s}$.

(b) $\mathrm{Shift}_{\vec{m}}(\mathcal{N}) \subset \mathcal{M}$ *is an inclusion of II_1 factors with index* $[\mathcal{M} : \mathcal{N}](\dim(X))^{2|\vec{m}|}$ *and with relative commutant* $\mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{M} \cong A_{|\vec{m}|}$.

(c) $\mathrm{Shift}_{\vec{m}}(\mathcal{N}) \subset \mathcal{N}$ *is an inclusion of II_1 factors with index* $(\dim(X))^{2|\vec{m}|}$ *and with relative commutant* $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M} \cong A_{m_1} \otimes \cdots \otimes A_{m_s}$.

Proof. For (a), we first show that the maps $\hat{\omega}_n$ in 3.5 define periodic commuting squares for $\mathrm{Shift}_{\vec{m}}(\mathcal{M}) \subset \mathcal{M}$. For this, one simply uses the fact that these maps are compositions involving the maps $\hat{v}_{\vec{m},n}$, $\hat{u}_{\vec{m},n}$ and $\hat{u}_n$ (see 3.6). They appear in the periodic commuting squares in Diagram 3.4, Diagram 3.2 and Diagram 3.3, see Lemma 3.2 and Lemma 3.4. Hence the desired diagram can be built from the just mentioned commuting squares. Periodicity is shown as in Lemma 3.2, and we can use the formula for the index, as done there. It follows from Lemma 3.1,(b) and (d), that the ratio of the square lengths of the weight vectors for A_{sn} and $A_{sn+|\vec{m}|}$ is equal to $(\dim X)^{2|\vec{m}|}$.

The statement about the relative commutant follows from the definition of $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}$. Let us represent $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(b)$, for $b \in A_{sn}$ ($s = 3$ to make things simpler) as it appears in Figure 3.11. Then for $a \in (t_{\vec{m}} \otimes 1_{sn})(A_{m_1} \otimes \cdots \otimes A_{m_s} \otimes 1_{sn})(t_{\vec{m}}^* \otimes 1_{sn}) \in A_{|\vec{m}|+sn}$ we have $a\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(b) = \mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(b)a$, which follows from Fig. 3.14. Hence $(t_{\vec{m}} \otimes 1_{sn})(A_{m_1} \otimes \cdots \otimes A_{m_s} \otimes 1_{sn})(t_{\vec{m}}^* \otimes 1_{sn}) \cong A_{m_1} \otimes \cdots \otimes A_{m_s}$ commutes with $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(b)$ for $b \in A_{sn}$, for every n, so that $\mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(\mathcal{M})' \cap \mathcal{M}$ has a subalgebra isomorphic to $A_{m_1} \otimes \cdots \otimes A_{m_s}$. This proves the last statement of (a).

For (b), one shows as before that the generating diagram for $\mathrm{Shift}_{\vec{m}}(\mathcal{N}) \subset \mathcal{M}$ is obtained by composing Diagram 3.4 and the square obtained from Diagram 3.6, which are both commuting. So Diagram 3.6 is a composition of those diagrams, and therefore is a periodic commuting square as well. The indices for parts (b) and (c) can now be computed as before, using Lemma 3.1. It only remains to show the statement about the relative commutant.

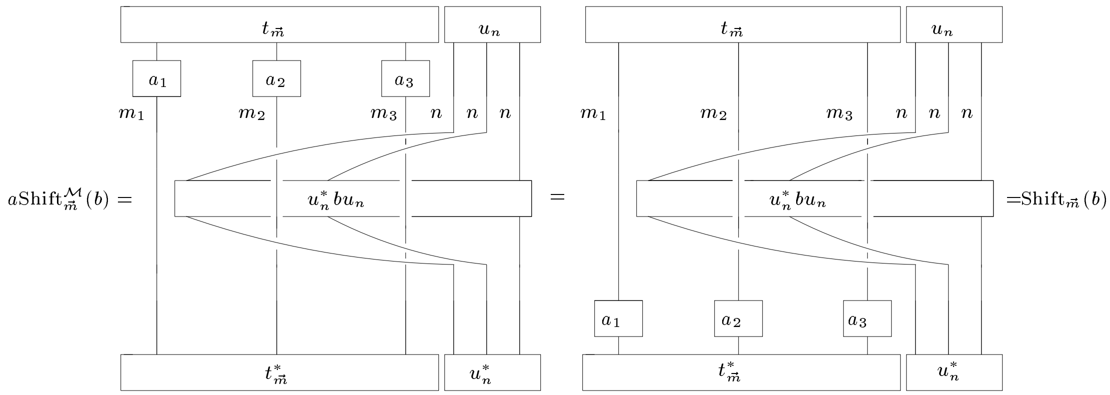

FIGURE 3.14

Lemma 3.5,(c), implies that $\mathrm{Shift}^{\mathcal{M}}_{\vec m}(u_n A_n^{\otimes s} u_n^*) = 1_{|\vec m|} \otimes u_n A_n^{\otimes s} u_n^*$ for every n. So $A_{|\vec m|} \otimes 1_{sn}$ commutes with $\mathrm{Shift}^{\mathcal{M}}_{\vec m}(u_n A_n^{\otimes s} u_n^*)$ for every n and $\mathrm{Shift}^{\mathcal{M}}_{\vec m}(\mathcal{N})' \cap \mathcal{M}$ has a subalgebra isomorphic to $A_{|\vec m|}$. Conversely, for the other inclusion, we apply a dimension upper bound result for relative commutants of inclusions generated by periodic commuting squares (see [W1], Theorem 1.6):

$$\begin{aligned}\dim\left(\mathrm{Shift}^{\mathcal{M}}_{\vec m}(\mathcal{N})' \cap \mathcal{M}\right) &\leq \dim\left((1_{|\vec m|} \otimes u_n A_n^{\otimes s} u_n^*)'_p \cap \left(A_{|\vec m|+sn}\right)_p\right)\\ &\leq \dim\left(A_{|\vec m|+sn}\right)_p,\end{aligned}$$

for any projection $p \in 1_{|\vec m|} \otimes u_n A_n^{\otimes s} u_n^*$, and n large. If n is divisible by k and sufficiently large, then $X^{\otimes n}$ contains a subobject isomorphic to $\mathbb{1}$; let $p_{\mathbf{1}} \in A_n$ be the projection onto it. If $p = 1_{|\vec m|} \otimes u_n(p_{\mathbf{1}}^{\otimes s})u_n^* \in A_{|\vec m|+ns}$, then we have $pA_{|\vec m|+ns}p \cong A_{|\vec m|}$. This shows (b).

For (c), it is even easier than in (a) to show that the generating Diagram 3.4 for $\mathrm{Shift}_{\vec m}(\mathcal{N}) \subset \mathcal{N}$ is a periodic commuting square; one can see that pictorially, as it was done in Lemmas 3.2 and Lemma 3.4, which is left to the reader. The statement about the relative commutant in (c) is proved in the same manner as in (b): By definition, $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(a_1 \otimes \cdots \otimes a_s) = (1_{m_1} \otimes a_1) \otimes \cdots \otimes (1_{m_s} \otimes a_s)$. Thus, $(A_{m_1} \otimes 1_n) \otimes \cdots \otimes (A_{m_s} \otimes 1_n)$ commutes with $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(A_n^{\otimes s})$ for every n, and so $\mathrm{Shift}^{\mathcal{N}}_{\vec m}(\mathcal{N})' \cap \mathcal{N}$ has a subalgebra isomorphic to $A_{m_1} \otimes \cdots \otimes A_{m_s}$. For the other inclusion we apply again the upper bound result for the dimension of the relative commutant:

$$\begin{aligned}\dim\left(\mathrm{Shift}^{\mathcal{N}}_{\vec m}(\mathcal{N})' \cap \mathcal{N}\right) &\leq \dim\left((1_{m_1} \otimes A_n) \otimes \cdots \otimes (1_{m_s} \otimes A_n)\right)'_p \cap (A_{n+m_1} \otimes \cdots \otimes A_{n+m_s})_p\\ &\leq \dim(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s})_p,\end{aligned}$$

for any projection $p \in (1_{m_1} \otimes A_n) \otimes \cdots \otimes (1_{m_s} \otimes A_n)$. One shows as in (b) that for $p = (1_{m_1} \otimes p_1) \otimes \cdots \otimes (1_{m_s} \otimes p_1)$ we have $(A_{n+m_1} \otimes \cdots \otimes A_{n+m_s})_p \cong A_{m_1} \otimes \cdots \otimes A_{m_s}$, from which one deduces (c). $\diamondsuit$

4. Bimodules and the principal graph

4.1. **Examples of bimodules.** We are going to construct systems of bimodules in order to calculate the principal and the dual principal graph, as described in Proposition 1.10. This will be done using the endomorphisms Shift defined in the last section.

The $\mathcal{N}$–$\mathcal{N}$-bimodules: Let $\lambda_i \in \Lambda$ and let A_{m_i,λ_i} be the simple component of A_{m_i} corresponding to the simple object $X_{\lambda_i} \subset X^{\otimes m_i}$ with m_i being large multiples of k for $i = 1, 2, \ldots, s$. We first fix minimal projections $p_{\lambda_i} \in A_{m_i,\lambda_i}$. Define $p_{\vec{\lambda}} = p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_s}$, where $\vec{\lambda} = (\lambda_1, \cdots, \lambda_s)$. The underlying Hilbert space will be given by

$$L^2(\mathcal{N}, tr)p_{\vec{\lambda}} := \{\zeta p_{\vec{\lambda}},\ \zeta \in L^2(\mathcal{N}, \mathrm{tr})\}.$$

The $\mathcal{N}$–$\mathcal{N}$ bimodule structure is defined by

$$x.\xi.y = x\xi \mathrm{Shift}_{\vec{m}}(y), \quad \text{for } x, y \in \mathcal{N},\ \xi \in L^2(\mathcal{N}, \mathrm{tr})p_{\vec{\lambda}},$$

where we use the usual right and left multiplication in $\mathcal{N}$ on the right hand side. It follows from Proposition 3.6 that this indeed defines an $\mathcal{N}$–$\mathcal{N}$ bimodules structure on $L^2(\mathcal{N}, tr)p_{\vec{\lambda}}$.

Definition 4.1. The $\mathcal{N}$–$\mathcal{N}$ bimodules defined above will be denoted by $N_{\vec{\lambda},\vec{m}}$.

The $\mathcal{M}$–$\mathcal{N}$-bimodules: Let again $\vec{m} := (m_1, \ldots, m_s) \in \mathbb{N}^s$, with $m := m_1 + \cdots + m_s$. We fix a minimal projection $p_\mu \in \mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{M} \cong A_m$ (see Proposition 3.6), where $\mu \in \Lambda$. The underlying Hilbert space for all these bimodules will be given by

$$L^2(\mathcal{M}, \mathrm{tr})p_\mu := \{\zeta p_\mu\ /\ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}.$$

The $\mathcal{M}$–$\mathcal{N}$ bimodule structure is defined by

$$x.\xi.y = x\xi \mathrm{Shift}_{\vec{m}}(y), \quad \text{for } x \in \mathcal{M},\ y \in \mathcal{N},\ \xi \in L^2(\mathcal{M}, \mathrm{tr})p_\mu.$$

Definition 4.2. The $\mathcal{M}$–$\mathcal{N}$-bimodules defined above will be denoted by $H_{\mu,\vec{m}}$.

The $\mathcal{N}$–$\mathcal{M}$-bimodules: With notations as in the last definition, we define similarly $\mathcal{N}$–$\mathcal{M}$-bimodules based on Hilbert spaces $p_\mu L^2(\mathcal{M}, \mathrm{tr}) := \{p_\mu \zeta\ /\ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}$, and with the $\mathcal{N}$–$\mathcal{M}$ bimodule structure defined by

$$x.\xi.y = \mathrm{Shift}_{\vec{m}}(x)\xi y, \quad \text{for } x \in \mathcal{N},\ y \in \mathcal{M},\ \xi \in p_\mu L^2(\mathcal{M}, \mathrm{tr}).$$

Definition 4.3. The $\mathcal{N}$–$\mathcal{M}$-bimodules defined above will be denoted by $K_{\mu,\vec{m}}$.

The $\mathcal{M}$–$\mathcal{M}$-bimodules: Similarly as for the $\mathcal{N}$–$\mathcal{N}$-bimodules, we fix minimal projections $p_{\lambda_i} \in A_{m_i,\lambda_i}$, with $\lambda_i \in \Lambda$, but now only requiring that $\sum m_i$ being divisible by k. The underlying Hilbert space for all these bimodules will be given by

$$p_{\vec{\lambda}} L^2(\mathcal{M}, tr) := \{p_{\vec{\lambda}}\zeta /\ \zeta \in L^2(\mathcal{M}, \mathrm{tr})\}.$$

The $\mathcal{M}$–$\mathcal{M}$ bimodule structure is defined by

$$x.\xi.y = \mathrm{Shift}_{\vec{m}}(x)\xi y, \quad \text{for } x, y \in \mathcal{M},\ \xi \in p_{\vec{\lambda}} L^2(\mathcal{M}, \mathrm{tr}),$$

Definition 4.4. The $\mathcal{M}$–$\mathcal{M}$-bimodules defined above will be denoted by $M_{\vec{\lambda},\vec{m}}$.

Lemma 4.5. *With the notation introduced above:*

(a) *If we view both $N_{\vec{\lambda},\vec{m}}$ and $H_{\nu,\vec{m}}$ as left $\mathcal{N}$-modules, then $\dim_{\mathcal{N}} N_{\vec{\lambda},\vec{m}} = d_{\vec{\lambda}}/(\dim X)^{|\vec{m}|}$ and $\dim_{\mathcal{N}} H_{\nu,\vec{m}} = d_\nu[\mathcal{M} : \mathcal{N}]/(\dim X)^{|\vec{m}|}$. Moreover, we have $\mathrm{ind}(N_{\vec{\lambda},\vec{m}}) = d_{\vec{\lambda}}^2$, where $d_{\vec{\lambda}} = \prod d_{\lambda_i}$, and $\mathrm{ind}(H_{\nu,\vec{m}}) = d_\nu^2[\mathcal{M} : \mathcal{N}]$.*

(b) *If $|\vec{m}| = |\vec{k}|$, then $H_{\mu,\vec{m}} \cong H_{\mu,\vec{k}}$ as $\mathcal{M}$–$\mathcal{N}$-bimodules, and $K_{\mu,\vec{m}} \cong K_{\mu,\vec{k}}$ as $\mathcal{N}$–$\mathcal{M}$-bimodules.*

(c) *If $|\vec{m}| = |\vec{k}|$, we have*

$$\mathrm{Hom}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda},\vec{m}}, M_{\vec{\mu},\vec{k}}) \subset \mathrm{Hom}_{\mathcal{N}-\mathcal{M}}(M_{\vec{\lambda},\vec{m}}, M_{\vec{\mu},\vec{k}}) \cong \mathrm{Hom}_{\mathcal{C}}(X_{\vec{\lambda}}, X_{\vec{\mu}}),$$

where $X_{\vec{\lambda}} = \otimes_{i=1}^s X_{\lambda_i}$ and $X_{\vec{\mu}} = \otimes_{i=1}^s X_{\mu_i}$.

Proof. We have the well-known facts that $\dim_{\mathcal{N}} L^2(\mathcal{N}, tr)p = tr(p)$ for any projection $p \in \mathcal{N}$, and $\dim_{\mathcal{N}} L^2(\mathcal{M}, tr)q = tr(q)[\mathcal{M} : \mathcal{N}]$ for any projection $q \in \mathcal{M}$ – see e.g. [Jo] – from which the dimension statements in (a) follow. For the index statements in (a), let ℓ and r denote left and right multiplication by $\mathcal{N}$ on $L^2(\mathcal{N}, tr)$ or suitable sub-modules of it. Observe that $\ell(\mathcal{N})'_{|L^2(\mathcal{N},tr)p}$ is equal to $r(p\mathcal{N}p)$ for any $p \in \mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{N}$. Recall that $\mathrm{Shift}_{\vec{m}}(\mathcal{N}) \subset \mathcal{N}$ has index $(\dim X)^{2|\vec{m}|}$. Moreover, $tr(p_{\vec{\lambda}}) = d_{\vec{\lambda}}/(\dim X)^{|\vec{m}|}$ for a minimal idempotent $p_{\vec{\lambda}} \in \mathrm{Shift}_{\vec{m}}(\mathcal{N})' \cap \mathcal{N}$, see Proposition 3.6. Using the formula for local indices, see [W1], Theorem 1.5,(iii), and the index formula in Proposition 3.6,(c), we obtain

$$\mathrm{ind}(N_{\vec{\lambda},\vec{m}}) = [p_{\vec{\lambda}}\mathcal{N}p_{\vec{\lambda}} : p_{\vec{\lambda}}\mathrm{Shift}_{\vec{m}}(\mathcal{N})] = tr(p_{\vec{\lambda}})^2(\dim X)^{2|\vec{m}|} = (d_{\vec{\lambda}})^2.$$

The index for $H_{\nu,\vec{m}}$ is computed similarly. By Lemma 3.5, (c), we have $\mathrm{Shift}_{\vec{m}}^{\mathcal{N}} = \mathrm{Shift}_{\vec{k}}^{\mathcal{N}}$, from which (b) follows.

Let ${}_{\vec{m}}L^2(\mathcal{M}, tr)$ be the Hilbert space $L^2(\mathcal{M}, tr)$ with $\mathcal{M}$–$\mathcal{M}$ bimodule structure $x.\xi.y = \mathrm{Shift}_{\vec{m}}^{\mathcal{M}}(x)\xi y$ for $x, y \in \mathcal{M}$ and $\xi \in L^2(\mathcal{M}, tr)$. Define ${}_{\vec{k}}L^2(\mathcal{M}, tr)$ similarly. These bimodules are isomorphic as $\mathcal{N}$–$\mathcal{M}$ bimodules, again by Lemma 3.5, (c), which combined with Lemma 3.5,(b), result in

$$\mathrm{Hom}_{\mathcal{M}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, tr), {}_{\vec{k}}L^2(\mathcal{M}, tr)) \subset \mathrm{Hom}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, tr), {}_{\vec{k}}L^2(\mathcal{M}, tr)) \cong$$

$$\cong \mathrm{End}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, tr)) \cong A_{|\vec{m}|} = \mathrm{End}_{\mathcal{C}}(X^{\otimes|\vec{m}|}), \qquad (*)$$

where the second isomorphism follows from Corollary 3.6,(b), and (b). By construction, we have $M_{\vec{\lambda},\vec{m}} = p_{\vec{\lambda}}({}_{\vec{m}}L^2(\mathcal{M}, tr))$ and $M_{\vec{\mu},\vec{k}} = (p_{\vec{\mu}}\,{}_{\vec{k}}L^2(\mathcal{M}, tr))$, where $p_{\vec{\lambda}} = p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_s}$ and $p_{\vec{\mu}} = p_{\mu_1} \otimes \cdots \otimes p_{\mu_s}$. Hence we can interpret an element $f \in \mathrm{Hom}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda}}, M_{\vec{\mu}})$ as an element in $\mathrm{Hom}_{\mathcal{N}-\mathcal{M}}({}_{\vec{m}}L^2(\mathcal{M}, tr), {}_{\vec{k}}L^2(\mathcal{M}, tr))$ which satisfies $p_{\vec{\mu}} f p_{\vec{\lambda}} = f$. Using this together with $(*)$ proves claim (c). $\diamondsuit$

4.2. **Principal graph.** Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_s) \in (\Lambda')^s$, and let $L^\nu_{\vec{\lambda}}$ be the multiplicity of the object X_ν in $\otimes X_{\lambda_i}$. Observe that $L^\nu_{\vec{\lambda}}$ is also equal to the rank of the projection $\bigotimes p_{\lambda_i}$ in the simple component of $A_{|\vec{\lambda}|}$ labeled by ν.

In the following we will fix a vector $\vec{m} = (m_i)$ where all its coordinates are divisible by k, and with m_i large enough that all simple objects of $\mathcal{C}'$ will appear in $X^{\otimes m_i}$ for $i = 1, \cdots, s$. We shall hence omit $\vec{m}$ in the indices of the bimodules and will just write $N_{\vec{\lambda}}$ and K_ν for $N_{\vec{\lambda},\vec{m}}$ and $K_{\nu,\vec{m}}$, respectively.

Theorem 4.6. *With the notation as above:*

(a) *The bimodules $N_{\vec{\lambda}}$ and H_ν defined above are irreducible.*
(b) *The principal graph for $\mathcal{N} \subset \mathcal{M}$ is the connected component of the fusion graph from $(\mathcal{C}')^s$ to $\mathcal{C}'$ which contains the trivial object of $\mathcal{C}$. Recall that the even vertices of the fusion graph are labeled by s-tuples of elements of Λ', the odd vertices are labeled by the elements of Λ', and the vertex labeled by $\vec{\lambda} = (\lambda_1, \cdots, \lambda_s)$ is connected with the vertex labeled by ν by $L^\nu_{\vec{\lambda}}$ edges.*
(c) *The subfactor $\mathcal{N} \subset \mathcal{M}$ has finite depth.*

Proof. Statement (a) follows from Proposition 3.6. It follows from the definitions that $L^2(\mathcal{M}, tr) \otimes_{\mathcal{N}} N_{\vec{\lambda}} \cong L^2(\mathcal{M}, tr)p_{\vec{\lambda}} \cong \oplus L^\nu_{\vec{\lambda}} H_\nu$; the decomposition of $L^2(\mathcal{M}, tr)p_{\vec{\lambda}}$ into irreducible $\mathcal{M}$–$\mathcal{N}$ bimodules follows from Proposition 3.6,(b) and the remarks at the beginning of this subsection. Hence our system of bimodules $(N_{\vec{\lambda}})_{\vec{\lambda} \in (\Lambda')^s}$ and $(H_\nu)_{\nu \in \Lambda'}$ is closed under induction. To prove closedness under restriction, observe that the multiplicity of the $\mathcal{N}$–$\mathcal{N}$ bimodule $N_{\vec{\lambda}}$ in the $\mathcal{M}$–$\mathcal{N}$-bimodule H_ν, viewed as an $\mathcal{N}$–$\mathcal{N}$-bimodule, is equal to $L^\nu_{\vec{\lambda}}$, by Frobenius reciprocity. To show that $H_\nu \cong \bigoplus_{\vec{\lambda}} L^\nu_{\vec{\lambda}} N_{\vec{\lambda}}$ as an $\mathcal{N}$–$\mathcal{N}$-bimodule, it suffices to prove that both sides have the same dimension, i.e., by Lemma 4.5,(a), that

$$[\mathcal{M} : \mathcal{N}]d_\nu = \sum_{\vec{\lambda}} L^\nu_{\vec{\lambda}} d_{\vec{\lambda}}. \tag{4.1}$$

For this observe that the dimension vectors for $A_n^{\otimes s}$ and A_{ns}, with n a multiple of k, are given by $\vec{t}_{ns} = (d_{\vec{\lambda}}/(\dim X)^{ns})_{\vec{\lambda}}$ and $\vec{v}_{ns} = (d_\nu/(\dim X)^{ns})_\nu$, with $\vec{\lambda} \in (\Lambda')^s$ and $\nu \in \Lambda'$. Observe that the subfactor $\mathcal{N} \subset \mathcal{M}$ is generated by the periodic sequence $(A_n^{\otimes s} \subset A_{ns})$, with the inclusion matrix for $A_n^{\otimes s} \subset A_{ns}$ given by $G = (L^\nu_{\vec{\lambda}})$ with $\vec{\lambda}$ and ν as above, provided $k|n$. Hence it follows from [W1], Theorem 1.5,(ii), that $G\vec{v}_{ns} = [\mathcal{M} : \mathcal{N}]\vec{t}_{ns}$. This implies Equation 4.1. Statement (c) is a consequence of (b). $\diamondsuit$

Remark 4.7. There are cases where the fusion graph from $(\mathcal{C}')^s$ to $\mathcal{C}'$ is not connected. An easy example is obtained for $\mathcal{C}$ being the representation category of a finite abelian group G, where it decomposes into $|G|$ connected components.

5. Dual principal graph

5.1. **Ring lemma.** The precise structure of $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ is still open after Proposition 3.6. To say more about this, we need the following lemma. Similar techniques have appeared before in topological quantum field theory, and within subfactors in work of Ocneanu and others, see e.g. [EK2], [M2].

Lemma 5.1. *If $a \in \mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$, take $\tilde{a} := t^*_{\vec{m}} a t_{\vec{m}}$ with $t_{\vec{m}} \in A_{|\vec{m}|}$ as in Figure 3.6. Then, the following relations hold for $r = 2, \ldots, s$:*

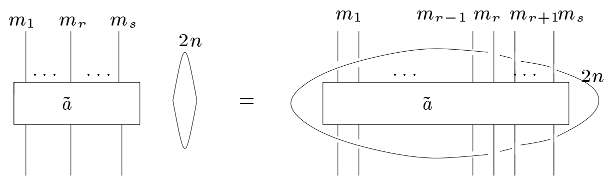

FIGURE 5.15. $\iota^r_X(\tilde{a} \otimes 1_{sn})\iota^r_X{}^* = \iota^r_X x_r(\tilde{a} \otimes 1_{sn}) x^*_r \iota^r_X{}^*$

where the morphisms x_r and ι^r_X will be defined below.

Proof. By Proposition 3.6, (b), we know that $\mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M} \subset A_{|\vec{m}|}$. Take $t_{\vec{m}} \in A_{|\vec{m}|}$ as in Figure 3.6. If $a \in \mathrm{Shift}_{\vec{m}}(\mathcal{M})' \cap \mathcal{M}$ then set

$$\tilde{a} \otimes 1_{sn} := (t^*_{\vec{m}} \otimes u^*_n) a (t_{\vec{m}} \otimes u_n) = t^*_{\vec{m}} a t_{\vec{m}} \otimes 1_{sn} \in A_{|\vec{m}|} \otimes 1_{sn},$$

and note that $\tilde{a} \otimes 1_{sn} \in \left((t^*_{\vec{m}} \otimes u^*_n)\mathrm{Shift}_{\vec{m}}(\mathcal{M})(t_{\vec{m}} \otimes u_n)\right)' \cap \mathcal{M}$. In particular, take the element $x_r := (t^*_{\vec{m}} \otimes u^*_n)\mathrm{Shift}_{\vec{m}}(u_n T_r u^*_n)(t_{\vec{m}} \otimes u_n)$, for $r = 2, \ldots, s$, where $T_r \in A_{sn}$ is obtained from the braiding morphisms and can be represented by the picture:

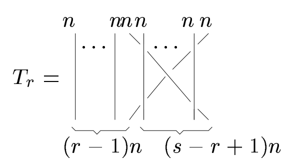

FIGURE 5.16

We use Figure 3.11 in the proof of Lemma 3.5 to see that x_r is given by Figure 5.17.

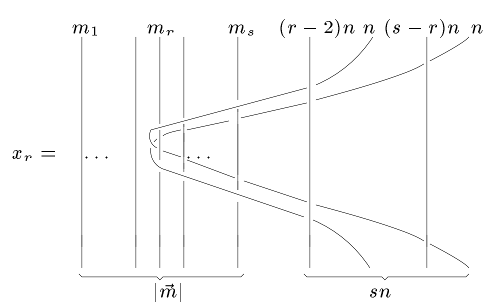

FIGURE 5.17. $x_r := (t^*_{\vec{m}} \otimes u^*_n)\mathrm{Shift}_{\vec{m}}(u_n T_r u^*_n)(t_{\vec{m}} \otimes u_n)$

Also note that x_r is a unitary, so that $(\tilde{a} \otimes 1_{sn})x_r = x_r(\tilde{a} \otimes 1_{sn})$ implies $(y\tilde{a} \otimes 1_{sn}) = x_r(\tilde{a} \otimes 1_{sn})x_r^*$. This is pictorially represented in Figure 5.18:

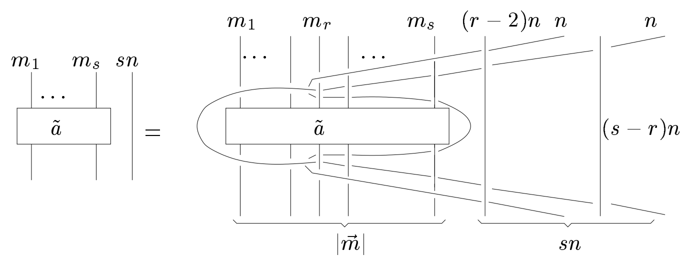

FIGURE 5.18. $(\tilde{a} \otimes 1_{sn}) = x_r(\tilde{a} \otimes 1_{sn})x_r^*$.

In order to obtain the relations in our statement in Figure 5.15, we proceed by "closing" strands in Figure 5.18 with "cups" and " caps" to form the loops (where the caps and cups correspond to dual morphisms as described in the subsection 2.2). This is done as follows: Let rh and lh be the left and right hand sides of Figure 5.18. Then we also obtain $rh \otimes 1_{(\bar{X})^{sn}} = lh \otimes 1_{(\bar{X})^{sn}}$. We now multiply both sides with $1_{X^{\otimes|\vec{m}|}} \otimes i_{X^{\otimes sn}}$ from the right (below) and by its conjugate from the left (above). The morphisms $i_{X^{\otimes sn}}$ and its conjugate correspond to the pictures in Figure 5.19, which are obtained from the properties of the duality morphisms, see Section 2.2. It is easy to check that we obtain $(s-2)n$ unlinked

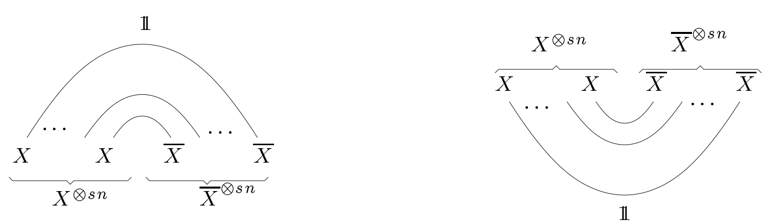

FIGURE 5.19. $\iota^*_{X^{\otimes(sn)}}$ and $\iota_{X^{\otimes(sn)}}$

circles on the right hand side, which correspond to the scalar $(\dim X)^{(s-2)n}$. Canceling this with the same number of circles on the left hand side, we obtain the picture as claimed in the statement. $\diamondsuit$

Corollary 5.2. *The equality in Lemma 5.1 still holds if the rings on both sides are labeled by an irreducible object in* $\mathcal{C}'$.

Proof. Assume that $k|n$. Then the proof of Lemma 5.1 works as well if we multiply T_r by $1_{(r-1)n} \otimes p_{\mathbf{1}} \otimes 1_{(s-r+1)n} \otimes p_\mu$ where $p_{\mathbf{1}}$ and p_μ are projections onto irreducible objects appearing in $X^{\otimes n}$ isomorphic to $\mathbb{1}$ and to X_μ, respectively. Going through the proof of Lemma 5.1, we obtain the statement of the corollary at the end. $\diamondsuit$

5.2. **Notations and preliminaries.** For any braided semisimple tensor category $\mathcal{C}$ we can define a scalar $s_{\lambda\mu} = Tr(c_{\mu,\lambda}c_{\lambda,\mu})$, where $c_{\lambda,\mu}$ is the braiding morphism for $X_\lambda \otimes X_\mu$. The S-matrix is then given by $(s_{\lambda\mu})$, where the rows and columns are labeled by the simple objects of $\mathcal{C}$.

Let now $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We define $\mathcal{T}_\mathcal{D}$ to be the set of simple objects X_λ in $\mathcal{D}$ for which $s_{\lambda\mu} = \dim(X_\lambda)\dim(X_\mu)$ for all simple objects X_μ in $\mathcal{C}'$. We will primarily be interested only in the cases $\mathcal{D} = \mathcal{C}$ and $\mathcal{D} = \mathcal{C}'$. We usually assume $\mathcal{D}$ to be fixed, in which case we may just write $\mathcal{T}$ for $\mathcal{T}_\mathcal{D}$.

Let $X = \oplus_\lambda m_\lambda X_\lambda$, $Y = \oplus n_\lambda X_\lambda$ be objects in $\mathcal{C}$, and let $f : X \to Y$ be a morphism. Then f can be written as $f = \oplus f_\lambda$, where $f_\lambda : m_\lambda X_\lambda \to n_\lambda X_\lambda$. For given $f : X \to Y$, we define the morphism $f_\mathcal{T} : X_\mathcal{T} \to Y_\mathcal{T}$, where $f_\mathcal{T} = \oplus_{X_\lambda \in \mathcal{T}} f_\lambda$, and $X_\mathcal{T}$, $Y_\mathcal{T}$ are defined accordingly. Also, we define $p_\mathcal{T}(X) \in \mathrm{End}(X)$ to be the projection from X onto $X_\mathcal{T}$.

For a fixed object Z in $\mathcal{C}$ and a morphism $f : X \to Y$ we define the morphism $P_Z(f) : X \to Y$ by the following picture:

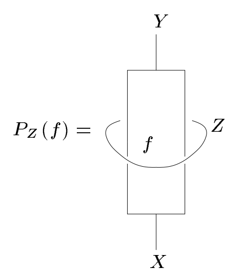

FIGURE 5.20

Of course this picture corresponds to an algebraic expression involving rigidity and braiding morphisms. One can also easily check that for $Z = Z_1 \otimes Z_2$, the operation P_Z is also given by a picture involving two parallel rings labeled by Z_1 and Z_2. Observe that if X_λ, X_μ are simple objects in $\mathcal{C}$, it follows from the definitions that $P_{X_\mu}(1_{X_\lambda}) = (s_{\lambda\mu}/d_\lambda)1_{X_\lambda}$. For a formal linear combination $\Omega = \sum_\mu \omega_\mu X_\mu$, with X_μ simple objects in $\mathcal{C}$, the morphism $P_\Omega(f)$ can also be expressed as the sum $\sum_\mu \omega_\mu P_{X_\mu}(f)$. The following lemma is well-known and follows from the definitions:

Lemma 5.3. *With notations above, we have* $P_{X_\mu}(f) = \sum_\lambda \frac{s_{\lambda\mu}}{d_\lambda} f_\lambda$ *and* $P_\Omega(f) = \sum_{\lambda,\mu} \omega_\mu \frac{s_{\lambda\mu}}{d_\lambda} f_\lambda$.

The following proposition is a straightforward generalization of the results in [Br], Lemma 1.3; its proof uses the same arguments as the ones used in the proofs of [Br], Lemma 1.2 and 1.3.

Proposition 5.4. *Fix the category* $\mathcal{D}$ *and let* $\mathcal{T} = \mathcal{T}_\mathcal{D}$. *There exists a linear combination* $\Omega = \sum_{\mu\in\Lambda'} \omega_\mu X_\mu$ *such that* $P_\Omega(f) = f_\mathcal{T}$ *for any morphism* f *in* $\mathcal{D}$. *Moreover,* $\sum_\mu \omega_\mu d_\mu = 1$.

Proof. By Lemma 5.3, we have to find scalars ω_μ, $\mu \in \Lambda'$ such that $\sum_{\mu\in\Lambda'} \omega_\mu \frac{s_{\lambda\mu}}{d_\lambda}$ is equal to 1 or 0 depending on whether $X_\lambda \in \mathcal{T}$ or not. Observe that the second statement will also follow from this as $s_{\lambda\mu} = d_\lambda d_\mu$ for $X_\lambda \in \mathcal{T}$.

To do so, pick an object $X = \bigoplus_{\lambda\in\Lambda(\mathcal{D})} m_\lambda X_\lambda$ in $\mathcal{D}$ with $m_\lambda \neq 0$ for all $\lambda \in \Lambda(\mathcal{D})$. Let z_λ denote the corresponding minimal idempotent in the center of End(X). Then $P_{X_\mu}(z_\lambda) = \frac{s_{\lambda\mu}}{d_\lambda} z_\lambda$. It also follows immediately by drawing pictures that $P_{Z_1\otimes Z_2}(f) = P_{Z_1}(P_{Z_2}(f))$ for any $f \in$ End(X) (see also the proof of [Br], Lemma 1.2). Hence we obtain a representation of the fusion algebra of $\mathcal{C}'$ on V, the $\mathbb{C}$-span of the idempotents $z_\lambda, \lambda \in \Lambda(\mathcal{D})$, with each P_{X_μ} acting via a diagonal matrix with respect to the basis of z_λ's. It follows from Lemma 5.3 that P_{X_μ} acts via the same scalar on the central idempotent z_λ as on $z_{\mathbf{1}}$, for all simple objects X_μ in $\mathcal{C}'$, if and only if $\lambda \in \mathcal{T}$. Hence the projection onto span$\{z_\lambda, X_\lambda \in \mathcal{T}\}$ is in the image of the fusion algebra, which is spanned by the P_{X_μ}'s. So we can find scalars ω_μ such that this projection is written as $\sum_{\mu\in\Lambda'} \omega_\mu P_{X_\mu}$. The claim follows from this. $\diamondsuit$

5.3. Let $f : \otimes_{i=1}^s X_{\lambda_i} \to \otimes_{i=1}^s X_{\mu_i}$ be a morphism. Then we define, for any $r = 1, \ ..., \ s$ the morphism $\hat{f}_r : \otimes_{i=r+1}^s X_{\lambda_i} \otimes \overline{X}_{\mu_i} \to \otimes_{i=1}^r \overline{X}_{\lambda_i} \otimes X_{\mu_i}$ using rigidity and braiding morphisms for suitable objects as indicated in Fig. 5.21; if $r = s$, the source of $\hat{f}_s$ is defined to be $\mathbb{1}$. E.g. we have $\hat{f}_1 = \alpha \circ (1_{\bar\lambda_1} \otimes f \otimes 1_{\bar\mu_2} \otimes \ ... \ \otimes 1_{\bar\mu_s}) \circ \beta$, for suitable morphisms α and β. We denote $\hat{f} = \hat{f}_s$.

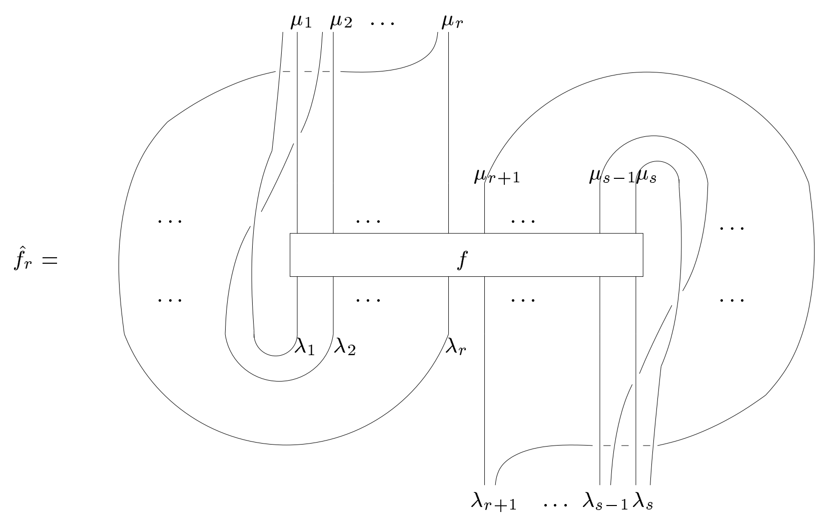

FIGURE 5.21. $\hat{f}_r : \otimes_{i=r+1}^s X_{\lambda_i} \otimes \overline{X}_{\mu_i} \to \otimes_{i=1}^r \overline{X}_{\lambda_i} \otimes X_{\mu_i}$

Corollary 5.5. *Let $f \in$ Hom$_{\mathcal{M}-\mathcal{M}}(M_{\vec\lambda}, M_{\vec\mu})$, viewed as an element in* Hom$_\mathcal{C}(X_{\vec\lambda}, X_{\vec\mu})$ *(see Lemma 4.5,(c)), and let P_Ω be as in Proposition 5.4. Then $\hat{f}_r = P_\Omega(\hat{f}_r) = (\hat{f}_r)_\mathcal{T}$.*

Proof. Fix r, and put a ring around f as it was done for $\tilde{a}$ in Lemma 5.1. By Corollary 5.2 the equality there also holds if we label the ring by $\Omega = \sum \omega_\mu X_\mu$, with the ω_μ as in Proposition 5.4. Observe that the ring on the left hand side becomes the scalar $\sum_\mu \omega_\mu d_\mu = 1$, by Proposition 5.4. Now multiply both sides with suitable morphisms which change f to

$\hat{f}_r$, such that all strands ending up go under the ring, and all strands ending at the bottom go above the ring. Then the right hand side is equal to $P_\Omega(\hat{f}_r)$, which is equalt to the left hand side $\hat{f}_r$. But by Proposition 5.4 $P_\Omega(\hat{f}_r) = (\hat{f}_r)_{\mathcal{T}}$.

Lemma 5.6. *If $f \in \mathrm{Hom}(M_{\vec{\lambda}}, M_{\vec{\mu}})$ then $\hat{f} = (\otimes_{i=1}^s p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}))\hat{f}$.*

Proof. We will prove by induction on r that $\hat{f}_r = \otimes_{i=1}^r p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}))\hat{f}_r$. For $r = 1$, we have

$$\hat{f}_1 = P_\Omega(\hat{f}_1) = (\hat{f}_1)_{\mathcal{T}},$$

by Corollary 5.5. This proves the claim for $r = 1$, as the target of the morphism $\hat{f}_1$ is $\bar{X}_{\lambda_1} \otimes X_{\mu_1}$. For the induction step we use the inductive formula for $\hat{f}_{r+1}$, as given in Figure 5.22.

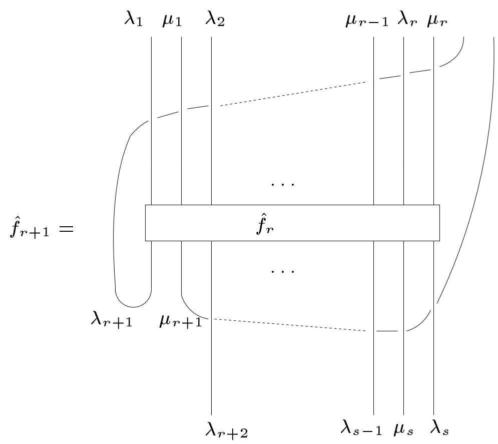

FIGURE 5.22

We obtain from this and the induction assumption that

$$\hat{f}_{r+1} = [(\otimes_{i=1}^r p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i})) \otimes 1_{\overline{X}_{\lambda_{r+1}} \otimes X_{\mu_{r+1}}}]\hat{f}_{r+1}.$$

Proceeding as in the case $r = 1$, we also obtain

$$\hat{f}_{r+1} = P_\Omega(\hat{f}_{r+1}) = p_{\mathcal{T}}(\otimes_{i=1}^{r+1} \overline{X}_{\lambda_i} \otimes X_{\mu_i}) P_\Omega(\hat{f}_{r+1}).$$

If X_λ is an object in $\mathcal{T}$, then so is $\bar{X}_\lambda$ (see remarks in the next subsection). It follows that the tensor product of simple objects $X_\lambda \otimes X_\mu$ is in $\mathcal{T}$ for $X_\lambda \in \mathcal{T}$ only if also X_μ is in $\mathcal{T}$. One deduces from this that

$$[(\otimes_{i=1}^r p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i})) \otimes 1_{\overline{X}_{\lambda_{r+1}} \otimes X_{\mu_{r+1}}}] p_{\mathcal{T}}(\otimes_{i=1}^{r+1} \overline{X}_{\lambda_i} \otimes X_{\mu_i}) = \otimes_{i=1}^{r+1} p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}).$$

This proves the claim. $\diamondsuit$

5.4. It can be shown under fairly weak conditions that the category $\mathcal{T}$ is equivalent to the representation category of a finite group G, see the papers [Br] and [M1]. In the following, we shall require in addition that $\mathcal{T}$ is equivalent to the representation category of a finite *abelian* group G, for any choice of $\mathcal{D}$. In this case, every simple object in the subcategory $\mathcal{T}$ is invertible. Moreover, we can and will label the simple objects of $\mathcal{T}$ by the elements of G in such a way that $X_g \otimes X_h \cong X_{gh}$ for any $g, h \in G$. Then we get a G-action on the index set Λ defined by $X_{g.\lambda} = X_g \otimes X_\lambda$. We shall also need the subgroup G_1^s of G^s consisting of all s-tuples $(g_1, g_2, \ldots, g_s)$ which satisfy $g_1 g_2 \cdots g_s = 1$. The just defined G-action extends to an action of G_1^s on Λ^s in the obvious way.

Proposition 5.7. *Under the above assumptions we have*

(a) $\mathrm{Hom}(M_{\vec{\lambda}}, M_{\vec{\mu}}) \neq 0$ *only if there exists a* $g \in G_1^s$ *such that* $\vec{\mu} = g.\vec{\lambda}$.
(b) $\dim \mathrm{End}(M_{\vec{\lambda}}) \leq |Stab_{G_1^s}\vec{\lambda}|$.

Proof. We use notations as in Lemma 5.6. By our assumptions, we have $p_{\mathcal{T}}(\overline{X}_{\lambda_i} \otimes X_{\mu_i}) = 0$ unless we can find an element $g_i \in G$ such that $X_{g_i} \subset \overline{X}_{\lambda_i} \otimes X_{\mu_i}$. This implies $g_i.\lambda_i = \mu_i$, and hence $\vec{\mu} = g.\vec{\lambda}$ for some $g \in G^s$. Moreover, we have a nonzero morphism from $\mathbb{1}$ to $\otimes X_{g_i}$ if and only if $\prod g_i = 1$. This shows that $g \in G_1^s$, by Lemma 5.6.

By the discussion in the previous paragraph, the dimension of $\mathrm{Hom}(\mathbb{1}, \otimes_i p_G(\overline{X}_{\lambda_i} \otimes X_{\lambda_i}))$ is equal to the cardinality of all s-tuples $g = (g_i)$ of elements of G for which $g.\vec{\lambda} = \vec{\lambda}$ and whose product $\prod g_i$ is equal to 1. These are exactly the elements of $\mathrm{Stab}_{G_1^s}\vec{\lambda}$. The claim now follows from the fact that the map $f \mapsto \hat{f}$ is injective; indeed, it is easy to construct a left-inverse by multiplying $\hat{f}$ by a suitable combination of $\cap$'s and $\cup$'s to get back f. $\diamondsuit$

Theorem 5.8. *If the S-matrix for the category $\mathcal{C}'$ is invertible, the dual principal graph for the inclusion $\mathcal{N} \subset \mathcal{M}$ coincides with its principal graph. In particular, each $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$, with $\vec{\lambda} = (\lambda_i)$ such that each λ_i labels a simple object in $\mathcal{C}'$ is irreducible.*

Proof. We will use the results of Lemma 5.6 and of Proposition 5.7 for the category $\mathcal{C}'$. If the S-matrix is invertible, G is the trivial group. Hence there are no nonzero morphisms between $M_{\vec{\lambda}}$ and $M_{\vec{\mu}}$ for $\vec{\lambda} \neq \vec{\mu}$, and each $\mathcal{M}$–$\mathcal{M}$-bimodule $M_{\vec{\lambda}}$ is irreducible by Proposition 5.7. It follows from the definitions (see before Theorem 4.6) that the multiplicity of a simple $\mathcal{N}$–$\mathcal{M}$ bimodule $\tilde{M}_\nu$ in the simple $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$ is equal to $L^\nu_{\vec{\lambda}}$.

Observe that $\mathrm{ind}(K_\nu) = d_\nu^2[\mathcal{M} : \mathcal{N}]$ and $\mathrm{ind}(M_{\vec{\lambda}}) = \prod_i d^2_{\lambda_i}$. It follows that

$$\sum_{\nu \in \Lambda'} d_\nu^2 [\mathcal{M} : \mathcal{N}] = (\sum_{\nu \in \Lambda'} d_\nu^2)^s = \sum_{\vec{\lambda} \in (\Lambda')^s} \prod_i d^2_{\lambda_i}.$$

Hence $\sum_{\nu \in \Lambda'} \mathrm{ind}(K_\nu) = \sum_{\vec{\lambda} \in (\Lambda')^s} \mathrm{ind}(M_{\vec{\lambda}})$. As any simple $\mathcal{N}$–$\mathcal{M}$-bimodule in a higher relative commutant is weakly isomorphic to an element in $(K_\nu)_{\nu \in \Lambda'}$, by Theorem 4.6, it follows from Lemma 1.12,(a), that there can not be any additional $\mathcal{M}$–$\mathcal{M}$-bimodules in the higher relative commutants. $\diamondsuit$

5.5. **Non-invertible S-matrix.** We shall make the following assumptions: We assume that the category $\mathcal{T}$ for our chosen category $\mathcal{D} = \mathcal{C}$ is equivalent to the representation category of a finite abelian group G, and, moreover, that $|G| = k$, with k as defined in Section 3.1. This also implies that $|G_1^s| = k^{s-1}$.

Theorem 5.9. *We assume the conditions stated at the beginning of this subsection. Then we have:*

(a) $\dim \mathrm{End}_{\mathcal{M}-\mathcal{M}}(M_{\vec{\lambda}}) = |Stab_{G_1^s}\vec{\lambda}|$ *for any* $\vec{\lambda} \in \Lambda_0^s := \{\vec{\lambda} \in \Lambda^s,\ k|\sum|\lambda_i|\}$.

(b) *The even vertices of the dual principal graph of the inclusion $\mathcal{N} \subset \mathcal{M}$ are labeled by the equivalence classes of irreducible components of the bimodules $M_{\vec{\lambda}}$, with $\vec{\lambda} \in \Lambda_0^s$.*

Proof. Let $M_{\vec{\lambda}} = \bigoplus_i Q_{\vec{\lambda},i}$ be the decomposition of the $\mathcal{M}$–$\mathcal{M}$ bimodule $M_{\vec{\lambda}}$ into irreducible $\mathcal{M}$–$\mathcal{M}$-bimodules. Then it follows from Lemma 1.12,(b), and Proposition 5.7 that

$$\sum_i \mathrm{ind}(Q_{\vec{\lambda},i}) \geq \frac{\mathrm{ind}(M_{\vec{\lambda}})}{\dim(\mathrm{End}(M_{\vec{\lambda}}))} \geq \frac{\mathrm{ind}(M_{\vec{\lambda}})}{|\mathrm{Stab}_{G_1^s}\vec{\lambda}|}.$$

Let now $(Q_j)_j = \bigcup_{\vec{\lambda}}(Q_{\vec{\lambda},i})_i$ be the collection of mutually nonisomorphic representatives of irreducible $\mathcal{M}$–$\mathcal{M}$ submodules of any module $M_{\vec{\lambda}}$ with $\vec{\lambda} \in \Lambda_0^s$. Then we have

$$\sum_j \mathrm{ind}(Q_j) \geq \sum_{G_1^s-orbits \in \Lambda_0^s} \frac{\mathrm{ind}(M_{\vec{\lambda}})}{|\mathrm{Stab}_{G_1^s}\vec{\lambda}|} =$$

$$= \frac{1}{k^{s-1}} \sum_{\vec{\lambda} \in \Lambda_0^s} \mathrm{ind}(M_{\vec{\lambda}}) = \frac{1}{k^{s-1}}(\frac{1}{k}\sum_{\lambda\in\Lambda} d_\lambda^2)^s = (\sum_{\lambda\in\Lambda'} d_\lambda^2)^s,$$

where we use Lemma 3.1,(d), for the last equality. But the sum $(\sum_{\lambda\in\Lambda'} d_\lambda^2)^s$ is equal to $\sum_{\mu\in\Lambda'} \mathrm{ind}(K_\mu)$, as was already shown in the proof of Theorem 5.8. Hence the inequalities above must be equalities, and our set of bimodules $(Q_j)_j$ must already exhaust all possible $\mathcal{M}$–$\mathcal{M}$-bimodules in the higher relative commutant, by Lemma 1.12. $\diamondsuit$

Remark 5.10. If the stabilizer $\mathrm{Stab}_{G_1^s}\vec{\lambda}$ is trivial, which usually is the case for most labels, the bimodule $M_{\vec{\lambda}}$ is irreducible, and its decomposition into $\mathcal{N}$–$\mathcal{M}$-bimodules is again determined by the fusion coefficients $L_{\vec{\lambda}}^{\nu}$. Unfortunately, our theorem does not say anything about what $\mathrm{End}(M_{\vec{\lambda}})$ looks like if $|\mathrm{Stab}_{G_1^s}\vec{\lambda}| \geq 4$. E.g., if the stabilizer has four elements, $\mathrm{End}(M_{\vec{\lambda}})$ could be isomorphic to $\mathbb{C}^4$ or to the 2×2 matrices. Neither does it say how the submodules of $\mathcal{M}_{\vec{\lambda}}$ decompose into irreducible $\mathcal{N} - \mathcal{M}$ modules in these cases.

6. Examples

6.1. **Examples of C^*-tensor categories.** 1. The easiest example for our set-up is the representation category $Rep(G)$ of finite dimensional unitary representations of a finite group. Here the braiding structure is just given by the permutation of tensor factors, which commutes with the group action. This makes the S-matrix a rank 1 matrix, i.e. noninvertible unless G is trivial. However, at least in principal, the dual principal graph can be computed from a general result about fixed point algebras of a group K and its subgroup H. In our setting, $K = G^s$ and $H \cong G$, which is embedded by $g \in G \mapsto (g, g, \cdots, g)$ (s times). See [KMY] for details.

In the special case when the subgroup K is normal, we obtain principal and dual principal graphs of the factor group H/K. This is the case in our setting if G is abelian.

2. Let ρ be a II_1 factor representation of the infinite braid group $\mathcal{B}_\infty$ such that the Jones index for the inclusion of factors $\rho(\mathcal{B}_{2,\infty})'' \subset \rho(\mathcal{B}_\infty)''$ is finite. Let us define $A_n = \rho(\mathcal{B}_{n+1,\infty})' \cap \rho(B_\infty)''$. We moreover assume that there exists, for some $k \in \mathbb{N}$, a projection $p \in A_k$ such that $p\rho(\mathcal{B}_\infty)''p = p\rho(B_{k+1,\infty})''$. It is possible to define from this a C^*-tensor category, with the objects being the projections in A_n. Most of this has already been done in [W2], Section 2, without mentioning categories. We shall not do this here. We just remark that the constructions of this paper will work in this setting without explicitly exhibiting the category; this has already been done in [E1]. In particular, this can be applied to the Jones subfactors as well as to the Hecke algebra and BCD type subfactors.

3. Let $U_q\mathfrak{g}$ be the Drinfeld-Jimbo deformation of the universal enveloping algebra $U\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$. It is well-known that the category of its finite dimensional representations has a braiding structure. It can not be unitarized except for $q = 1$. If q is a root of unity $\neq 1$, one can define a special class of representations called tilting modules which again forms a braided tensor category. It can be shown that the category of tilting modules has a semisimple quotient with only finitely many simple modules up to equivalence; this is often referred to as a fusion category (see [A],[AP]). Moreover, for q being certain roots of unity (usually of the form $q = e^{\pm 2\pi i/l}$ for suitable integers l (see [W3] for precise values), this quotient can be unitarized. This yields a large and important class of C^* tensor categories. Using the one-sided subfactor construction, one obtains the Jones subfactors for X being the $U_q sl_2$-analog of the 2-dimensional representation of sl_2. Similarly, Hecke algebra subfactors and BCD type subfactors can be obtained from fusion categories of quantum groups of classical Lie types.

These C^*-fusion categories can also be obtained by a completely different construction using the category of positive energy representation of a loop group. The difficulty in this construction comes from the fact that one can not use the usual tensor product for representations; instead one has to define a new, so-called fusion tensor product (see [Wa]).

4. Let $N \subset M$ be an inclusion of II_1 factors with finite index and finite depth. Then the category of N–N bimodules obtained as direct sums of summands of the bimodules $M^{\otimes n} = M \otimes_N M \otimes_N \cdots \otimes_N M$ (n times), $n \in \mathbb{N}$ defines a C^*-tensor category which may or may not be braided. One can similarly also define the C^*-tensor category of M-M bimodules generated by $M^{\otimes n}$.

If these categories are not braided, one can apply a general construction, called the categorical quantum double construction to construct from our category of bimodules a larger braided C^* tensor category. It was shown that this category is equivalent to the category of $\mathcal{M}$–$\mathcal{M}$-bimodules for the asymptotic inclusion $\mathcal{N} \subset \mathcal{M}$ derived from $N \subset M$, see [M2]. If the original category already was braided, the asymptotic inclusion coincides with the 2-sided inclusion constructed in this paper.

5. Our constructions of bimodules in this paper are based on certain endomorphisms of II_1 factors. The approach to categories via endomorphisms has been used for a long time for type III factors in the framework of algebraic quantum field theory (see e.g. [LR], [FRS], [X]). Here subtleties involving coupling constants do not matter, and objects are given directly by morphisms.

6.2. **Examples for our construction.** 1. Let us first list examples of C^*-tensor categories with invertible S-matrix.

(a) The S-matrix for the full fusion tensor categories as constructed in [A],[AP] is invertible under the conditions for unitarizability, as stated in [W3]. Hence if we can find an object X such that *all* irreducible representation of the fusion category appear in some tensor power of X, we have $\mathcal{C}' = \mathcal{C}$ and the dual principal graph is equal to the principal graph. Such representations can be found in all cases, but usually can not be chosen to be irreducible. E.g., for Lie type A (the case of Jones subfactors and Hecke algebra subfactors), one can choose $X = \mathbb{1} \oplus V$, where V is the analog of the vector representation.

(b) Similarly, the S-matrix for the quantum double of a C^* tensor category is always invertible (see e.g. [M2]). Hence, as soon as we have found an object X for which all irreducible representations of the double category appear in some tensor power of X, the dual principal graph of our s-sided inclusion with respect to X is equal to the principal graph.

2. It turns out that our construction does not only depend on the category $\mathcal{C}$, but also on the choice of the object X. Even though in the case of the fusion tensor categories the S-matrix for $\mathcal{C}$ is invertible, the S matrix for the category $\mathcal{C}'$ may not be invertible. E.g., for type A if one takes $X = V$, the S-matrix for $\mathcal{C}'$ is invertible only if the degree of the root of unity is coprime to k. If this is not the case, however, our results for noninvertible S-matrices apply. This will be shown in more detail in the following subsection at an example.

6.3. **Subfactors related to Jones subfactors.** We illustrate our examples in some detail for the fusion category $\mathcal{C}$ of $U_q sl_2$, with $q = e^{2\pi i/l}$. There also exist other, more elementary methods to construct these categories using the Temperley-Lieb algebras, see e.g. the book [T]. As mentioned before, this is also one of the cases where the subfactor constructions can be done on the level of braid representations, as it was carried out in the original paper [E1].

We give a brief description of this category. Up to isomorphism, we have exactly $l-1$ simple objects in $\mathcal{C}$, which are denoted by $[i]$, $1 \leq i \leq l-1$. The decomposition of tensor products is given by

$$[i] \otimes [j] = [|i-j|-1] \oplus [|i-j|+1] \oplus \cdots \oplus [m], \tag{6.1}$$

where m is the minimum of $i+j-1$ and $2l-1-i-j$. One sees easily that [1] corresponds to the trivial object. It follows from the tensor product rules by induction on n that all simple objects in $[2]^{\otimes n}$ are labeled by even numbers if n is odd, and by odd numbers if n is even. Hence $k = 2$ and the simple objects of $\mathcal{C}'$ are labeled by odd numbers. This explicitly describes the principal graph for $\mathcal{N} \subset \mathcal{M}$, constructed with $X = [2]$, by Theorem 4.6.

Observe that $[i] \otimes [l-1] = [l-i]$ for all $1 \leq i < l$. Hence the objects $[1]$ and $[l-1]$ together with the operation $\otimes$ form a group G which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the S matrix is well-known to be of the form $S = (\sin(ij\pi/l))$, up to a scalar.

It is very easy to check that if l is even and j is odd, then $\sin(i(l-1)\pi/l)) = \sin((i\pi/l))$ for any $i = 1, 2, \dots, l-1$. Hence the category $\mathcal{T}$ contains at least the objects $[1]$ and $[l-1]$. It contains no more simple objects as obviously $\sin(i\pi/l) = \sin(ij\pi/l)$ for $1 < j < l$ only if $j = l-1$. So the conditions at the beginning of Subsection 5.2 are satisfied with $|G| = 2 = k$. We have shown most of the following

Proposition 6.1. *Let $\mathcal{N} \subset \mathcal{M}$ be the subfactor constructed from the s-sided inclusion from the Jones subfactor at an l-th root of unity, with l even. Then we have*

(a) *The even vertices of the principal graph are labeled by all s-tuples of odd positive numbers less than l and the odd vertices are labeled by all odd positive numbers less than l. The number of edges between two vertices can be computed from the tensor product rule stated in 6.1.*

(b) *Each s-tuple of positive integers less than l whose sum is even and which contains the number $l/2$ at most once labels an even vertex of the dual principal graph; the number of edges emanating from such a vertex can be computed as in (a). The $\mathcal{M}$–$\mathcal{M}$ bimodules $M_{\vec{\lambda}}$ labeled by an s-tuple $\vec{\lambda}$ containing the number $l/2$ exactly $r > 1$ times satisfies $\dim(\mathrm{End}(M_{\vec{\lambda}})) = 2^{r-1}$.*

Proof. Part (a) follows from Theorem 4.6 and our explicit description of the simple objects of $\mathcal{C}'$. For part (b), we have already checked the conditions stated at the beginning of Subsection 5.2. It remains to calculate $\mathrm{Stab}_{G_1^s}\vec{\lambda}$ for any $\vec{\lambda} \in \Lambda^s$. Recall that the action of the nontrivial element of G on our labeling set is given by $i \mapsto l-i$. Obviously, the only fixed point is $l/2$ for l odd. It is now not hard to show that $\vec{\lambda} \in \Lambda^s$ has a nontrivial stabilizer in G_1^s if and only if $r \geq 2$ of its components are equal to $l/2$, and that in this case the stabilizer has exactly 2^{r-1} elements. Statement (b) now follows from Theorem 5.9. $\diamondsuit$

Remark 6.2. If $s = 3$, part (b) of the last proposition completely determines the number of edges in the dual principal graph except for the decomposition of the bimodule $M_{\vec{\lambda}}$ with $\vec{\lambda} = (l/2, l/2, l/2)$, which could decompose into the direct sum of four nonisomorphic irreducible $\mathcal{M}$–$\mathcal{M}$ bimodules or into the direct sum of two isomorphic irreducible $\mathcal{M}$–$\mathcal{M}$ bimodules.

References

[A] Andersen, Henning Haahr Tensor products of quantized tilting modules. Comm. Math. Phys. 149 (1992), no. 1, 149–159.

[AP] Andersen, Henning Haahr; Paradowski, Jan Fusion categories arising from semisimple Lie algebras. Comm. Math. Phys. 169 (1995), no. 3, 563–588.

[As] Asaeda, M., private communication.

[BWs] Barrett, John W.; Westbury, Bruce W. Spherical categories. Adv. Math. 143 (1999), no. 2, 357–375.

[BW] J. Birman; H. Wenzl, Braids, link polynomials and a new algebra, Trans. AMS **313** (1989) 249-273.

[Br] Bruguières, A., Catégories prémodulaires, modularisations et invariants des varieétés de dimension 3, Math. Ann. **316**, (2000), 215-236.

[Bs] Bisch, Dietmar, Bimodules, higher relative commutants and the fusion algebra associated to a subfactor. Operator algebras and their applications (Waterloo, ON, 1994/1995), 13–63, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.

[E1] Erlijman, J., Multi-sided braid type subfactors, Canad. J. Math. 53 (2001), 546-564.

[E2] Erlijman, J., Multi-sided braid type subfactors. II. Canad. Math. Bull. 46 (2003), no. 1, 80-94.

[EK1] Evans, David E.; Kawahigashi, Yasuyuki Quantum symmetries on operator algebras. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[EK2] Evans, David E.; Kawahigashi, Yasuyuki Orbifold subfactors from Hecke algebras. II. Quantum doubles and braiding. Comm. Math. Phys. 196 (1998), no. 2, 331–361.

[FRS] Fredenhagen, K.; Rehren, K.-H.; Schroer, B. Superselection sectors with braid group statistics and exchange algebras. I. General theory. Comm. Math. Phys. 125 (1989), no. 2, 201–226.

[Fr] Freyd, P., Abelian categories. An introduction to the theories of functors, Harper & Rows Publisher, New York 1964

[Iz] Izumi, Masaki The structure of sectors associated with Longo-Rehren inclusions. I. General theory. Comm. Math. Phys. 213 (2000), no. 1, 127–179.

[Jo] Jones, V.F.R., Index for subfactors, Invent. Math 72 (1983) 1–25.

[Ks] Kassel, Ch., Quantum groups, Springer Verlag 1995

[KLM] Kawahigashi, Yasuyuki; Longo, Roberto; Mger, Michael Multi-interval subfactors and modularity of representations in conformal field theory. Comm. Math. Phys. 219 (2001), no. 3, 631–669.

[KMY] Kosaki, Hideki; Munemasa, Akihiro; Yamagami, Shigeru On fusion algebras associated to finite group actions. Pacific J. Math. 177 (1997), no. 2, 269–290.

[LR] Longo, R.; Roberts, J. E. A theory of dimension. K-Theory 11 (1997), no. 2, 103–159.

[ML] MacLane, S. Categories for the working mathematician, Springer Verlag

[M1] Müger, Michael Galois theory for braided tensor categories and the modular closure. Adv. Math. 150 (2000), no. 2, 151-201.

[M2] Müger, Michael From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. J. Pure Appl. Algebra 180 (2003), no. 1-2, 159–219.

[OW] Orellana, R. C.; Wenzl, H. G. q-centralizer algebras for spin groups. J. Algebra 253 (2002), no. 2, 237–275.

[T] Turaev, V, Quantum invariants of knots and 3-manifolds, de Gruyter, Berlin, 1994

[Wa] Wassermann, Antony Operator algebras and conformal field theory. III. Fusion of positive energy representations of LSU(N) using bounded operators. Invent. Math. 133 (1998), no. 3, 467–538.

[W1] Wenzl, H., Hecke algebras of type A_n and subfactors, Invent. Math. 92 (1988) 349-383.

[W2] Wenzl, H.. Braids and invariants of 3-manifolds, Invent. math. 114 (1993) 235-275.

[W3] Wenzl, Hans C^* tensor categories from quantum groups. J. Amer. Math. Soc. 11 (1998), no. 2, 261–282.

[X] Xu, Feng Jones-Wassermann subfactors for disconnected intervals. Commun. Contemp. Math. 2 (2000), no. 3, 307–347.

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