COXETER CONSTRUCTION FOR HECKE ALGEBRAS

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Abstract. We give a generalization of Coxeter's construction of representations of reflection groups to braid groups.

It is well-known that one can associate to every graph X a braid group $B(X)$ as follows: the generators σ_i are labeled by the vertices i of the graph. The relations are given by $\sigma_i\sigma_j\sigma_i \ldots = \sigma_j\sigma_i\sigma_j \ldots$, where we have exactly $m(i,j)$ factors on each side of the equation, and where $m(i,j)$ is equal to 2+ the number of edges connecting i with j. While it is in general hard to decide whether generators and relations produce a nontrivial object, this is easy here thanks to Coxeter's geometric representation. Each σ_i is represented by a reflection, and one obtains a nontrivial representation of the braid group as a reflection group, i.e. where the generators also satisfy the relation $\sigma_i^2 = 1$.

The purpose of this note is to show that this construction can be easily generalized to a representation without the reflection property. We find continuous deformations of the reflection planes, depending on one or several parameters, which preserve the braid relations, but no longer the reflection property. Now the images T_i of the generators σ_i satisfy the Hecke relation $(T_i - q_i)(T_i + 1) = 0$, with q_i being a parameter. This representation can be defined in analogy of the Coxeter representation via a bilinear form, or also via a sesquilinear form; here the involution is given by $\bar{q}_i = q_i^{-1}$. Doing this construction over the complex numbers, we also determine for which values of q_i these representations can be unitarized, using a simple Gram-Schmid procedure.

For Coxeter graphs A_n, $n \in \mathbb{N}$, this Gram-Schmidt procedure leads to inductive formulas for a certain central idempotent of the corresponding Hecke algebras. These formulas already appeared in previous work [W1]. They found subsequently several applications in von Neumann algebras, mathematical physics and in topology. Unfortunately, one can not find such simple formulas for other graphs, i.e. the formula would essentially only hold for the given Coxeter representation. As another application, we give a fairly simple proof that a Hecke algebra has a natural basis labeled by the elements of the corresponding reflection group.

The perhaps most interesting consequence of this work is the occurrence of formal characters whose multiplication structure coefficients are q-version of the usual Clebsch-Gordan rules. We also obtain q-versions of $\cos(\pi/m)$ which so far are somewhat mysterious at least to this author. It would be quite interesting if a more conceptual explanation of these phenomena could be found.

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1. Coxeter construction

It is not the intention of this note to explore the utmost generality of this construction. We will assume K to be a field, with variables q_i. We shall freely adjoin algebraic functions in the variables q_i over this field whenever necessary.

1.1. **Hecke algebras.** Let X be a finite graph, with vertices labelled by the set S. Let, for the vertices labeled by i and j, the number $m(i,j)$ be equal to 2+ the number of edges between the vertices labelled by i and j. We fix a variable q_i for each vertex i with the condition $q_i = q_j$ if $m(i,j)$ is odd. The Hecke algebra $H = H(X;(q_i))$ over K is given by generators $T_i,\ i \in S$ and relations

(B) $T_iT_jT_i\ ...\ = T_jT_iT_j\ ...$, with $m(i,j)$ factors on each side,
(H) $T_i^2 = (q_i-1)T_i - q_i$.

1.2. Let q_1, q_2, θ be variables, and let χ_1 be a function in these variables such that $\chi_1(q_1,q_2;\theta) = \chi_1(q_2,q_1;\theta)$. We define formal 'characters' $\chi_n = \chi_n(q_1,q_2;\theta)$, $n \in \mathbf{Z}$ recursively by $\chi_n = 0$ for $n \le 0$, $\chi_0 = 1$ and

$$\chi_n\chi_1 = \begin{cases} \chi_{n+1} + q_1\chi_{n-1} & \text{if } n \text{ is odd,} \\ \chi_{n+1} + q_2\chi_{n-1} & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

So all functions χ_n are uniquely determined by χ_1. We also define the functions $\hat{\chi}_n$ by $\hat{\chi}_n(q_1,q_2;\theta) = \chi_n(q_2,q_1;\theta)$. We have $\hat{\chi}_1 = \chi_1$ by definition. It is easy to check that the functions $\{\chi_n,\ n \in \mathbb{N}\}$ span the algebra generated by χ_1, and that the operation $\hat{}$ defines an endomorphism of this algebra.

Example : If $q_1 = q_2 = q$, it is easy to check that the functions

$$\chi_n(q,q;\theta) = 2q^{n/2}\sin(n\theta)/\sin(\theta)$$

satisfy the conditions above.

Lemma 1.1. *$\hat{\chi}_n = \chi_n$ for n odd, and $\hat{\chi}_n + q_2\hat{\chi}_{n-2} = \chi_n + q_1\chi_{n-2}$ for n even. Moreover, we also have*

$$\chi_n\chi_2 = \begin{cases} \chi_{n+2} + q_2\hat{\chi}_{n-1}\chi_1 & \textit{if } n \textit{ is odd,} \\ \chi_{n+2} + q_2\chi_{n-1}\chi_1 & \textit{if } n \textit{ is even.} \end{cases} \tag{1.2}$$

Proof. The statements are proved by induction on n, with everything trivially true for $n \le 1$. Observe that

$$\chi_1^2\chi_n = \chi_{n+2} + (q_1+q_2)\chi_n + q_1q_2\chi_{n-2}. \tag{$*$}$$

Solving for χ_{n+2}, we obtain $\chi_{n+2} = \hat{\chi}_{n+2}$ for n odd by induction. In particular, $\hat{\chi}_1\hat{\chi}_n = \chi_1\chi_n$ for n odd, from which one derives the first statement for $\hat{\chi}_n$ with n even. It remains to prove Eq. 1.2. Using $(*)$ and $\chi_2 = \chi_1^2 - q_1$, we get, for n odd,

$$\chi_2\chi_n \ = \ \chi_{n+2} + q_2(\chi_n + q_1\chi_{n-2}) = \chi_{n+2} + q_2(\widehat{\chi_1\chi_{n-1}}).$$

The claim follows from $\hat{\chi}_1 = \chi_1$. The case for n even goes similarly.

1.3. We define matrices A_1 and A_2 by

$$A_1 = \begin{pmatrix} -1 & -\chi_1 \\ 0 & q_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} q_2 & 0 \\ -\chi_1 & -1 \end{pmatrix}.$$

Using the formulas in Lemma 1.1, it is easy to check by induction on k that

$$(A_1A_2)^kA_1 = \begin{pmatrix} -\hat{\chi}_{2k} & -\chi_{2k+1} \\ q_1\chi_{2k-1} & q_1\chi_{2k} \end{pmatrix}, \tag{1.3}$$

$$A_2(A_1A_2)^k = \begin{pmatrix} q_2\hat{\chi}_{2k} & q_2\chi_{2k-1} \\ -\chi_{2k+1} & -\chi_{2k} \end{pmatrix} \tag{1.4}$$

and

$$(A_1A_2)^k = \begin{pmatrix} \hat{\chi}_{2k} & \chi_{2k-1} \\ -q_1\chi_{2k-1} & -q_1\chi_{2k-2} \end{pmatrix} \tag{1.5}$$

Lemma 1.2. *The matrices A_1 and A_2 satisfy the identity $A_1A_2A_1 \ ... \ = A_2A_1A_2 \ ...$ (m factors on both sides) if and only if $\chi_{m-1} = 0$, and, if m is odd, $q_1 = q_2$.*

Proof. Assume $m = 2k+1$ and $\chi_{2k} = 0$. Then we also have $0 = \chi_1\chi_{2k} = \chi_{2k+1} + q_2\chi_{2k-1}$. The claimed identity now follows from this and formulas 1.3 and 1.4 for $m = 2k+1$. If $m = 2k$, observe that $PA_1P = \hat{A}_2$ and $PA_2P = \hat{A}_1$, where P is the 2×2 matrix permuting the two basis vectors. Hence $(A_2A_1)^k = P(\widehat{A_1A_2})^kP$. If $\chi_{2k-1} = 0$, we show as before that $\chi_{2k} = q_1\chi_{2k-2}$. It follows from the last two sentences that $(A_1A_2)^k = (A_2A_1)^k$ if $\chi_{2k-1} = 0$.

On the other hand, if $A_1A_2... = A_2A_1...$ (m factors in each side), then it follows from Eq 1.3-1.5 that $\chi_{m-1} = 0$ (as $q_1 \neq 1 \neq q_2$).

1.4. As suggested by the last lemma, we are not so much interested in the actual computation of the functions χ_n; it will be more important to compute the possible values of χ_1 for which χ_{m-1} will be equal to 0, for a given m.

Lemma 1.3. *Assume $q_1 = q_2 = q$. Then $\chi_{m-1} = 0$ if and only if $\chi_1 = q^{1/2}2\cos(j\pi/m)$, $j = 1, 2, \ ... \ m-1$.*

Proof. It is clear for the functions χ_m as defined at the end of Section 1.2 that $\chi_{m-1}(q, q; \theta) = 0$ if $\theta = j\pi/m$. Observe that on the other hand the values of $\chi_m(q_1, q_2; \theta)$ are already uniquely determined by the ones of $\chi_1(q_1, q_2; \theta)$. More precisely, if we set $\chi_1(q_1, q_2; \theta) = x$, then it follows from the recursion relation (1.1) that $\chi_{m-1}(q_1, q_2; \theta)$ is a polynomial in x, q_1 and q_2 of degree $m-1$ (in x). Hence there exist exactly $m-1$ values of x for which $\chi_{m-1}(q_1, q_2; \theta) = 0$.

Definition 1.4. Let $m \in \mathbb{N}$. We define $2\mathrm{co}(q_1, q_2; j\pi/m) = (q_1q_2)^{-1/4}x$, where $x = \chi_1$ is the solution in the equation $\chi_{m-1} = 0$ which specializes to $q^{1/2}2\cos(j\pi/m)$ for $q_1 = q_2 = q$.

Unfortunately, we do not have a nice formula for $2\mathrm{co}(q_1, q_2; j\pi/m)$, or, for that matter, an interpretation as a function. Nevertheless, it is easy to compute the χ_m's as polynomials in $x = \chi_1$, q_1 and q_2 from the recursion relation 1.1, as well as their zeros, for small m. We obtain $\chi_2 = x^2 - q_1$, $\chi_3 = x^3 - (q_1+q_2)x$ and $\chi_5 = x^5 - 2(q_1+q_2)x^3 + (q_1^2 + q_1q_2 + q_2^2)x$.

One obtains from this the following result:

$\chi_2 = 0: \quad \chi_1 = \pm\sqrt{q_1},$
$\chi_3 = 0: \quad \chi_1 = \pm\sqrt{q_1+q_2}$ or $\chi_1 = 0,$
$\chi_5 = 0: \quad \chi_1 = \pm\sqrt{q_1+q_2\pm\sqrt{q_1q_2}}$ or $\chi_1 = 0.$

1.5. Let V be a vector space with basis $(\alpha_i,\ i\in S)$, and with a bilinear form defined by

$$\langle\alpha_i,\alpha_i\rangle = 1+q_i \quad \text{and} \quad \langle\alpha_i,\alpha_j\rangle = -2(q_1q_2)^{1/4}\mathrm{co}(q_1,q_2;k\pi/m(i,j)) \quad \text{for } i\neq j,$$

where $\mathrm{co}(q_1,q_2;\pi/m(i,j))$ is as defined in the last section, and k is an integer satisfying $1\le k\le m(i,j)$. We also define for $i\in S$ the operator $T_i: V\to V$ by

$$T_iv = q_iv - \langle v,\alpha_i\rangle\alpha_i. \tag{1.6}$$

Observe that for $i\neq j\in S$, the operators T_i and T_j leave invariant the 2-dimensional subspace spanned by α_i and α_j. Their action on this subspace is described by the matrices A_i and A_j obtained by substituting q_1 by q_i and q_2 by q_j in the formulas for the matrices A_1 and A_2.

Theorem 1.5. *The Eq. 1.6 defines a representation of the Hecke algebra $H(X)$.*

Proof. Let $i\in S$. It follows from Eq.1.6 that $T_i\alpha_i = -\alpha_i$. Morover, $T_iv - q_iv\in K\alpha_i$, hence T_i acts via multiplication by q_i on the quotient space $V/K\alpha_i$. Hence T_i satisfies the equation $(T_i+1)(T_i-q_1) = 0$, which is equivalent to (H).

Let $i,j\in S$, and assume $m(i,j)<\infty$. Then T_i and T_j leave invariant the subspace W spanned by α_i and α_j, and act on W via the matrices A_i and A_j. These matrices satisfy relation (B), by Lemma 1.4. It is easy to check that the restriction of $\langle\ ,\ \rangle$ to W is nondegenerate for $m(i,j)<\infty$; this is easiest done by applying Gram-Schmid to the vectors α_i and α_j (see next section for more details). Hence $V = W\oplus W^\perp$. Both T_i and T_j act by multiplication by q_i resp. q_j on $W^\perp$, hence relation (B) is also satisfied on $W^\perp$.

If $m(i,j)=\infty$, the braid relation becomes void, and there is nothing to show.

2. Unitarizibility

2.1. **Bilinear Form.** We first show that the generators are self-adjoint with respect to our bilinear form $\langle\ ,\ \rangle$.

Lemma 2.1. *With notations as in the last section, we have $\langle T_iv,w\rangle = \langle v,T_iw\rangle$ for $v,w\in V$ and $i=1,2,\ \ldots\ n$.*

Proof. One computes $\langle T_iv,w\rangle = q_i\langle v,w\rangle + \langle v,\alpha_i\rangle\langle w,\alpha_i\rangle = \langle v,T_iw\rangle$.

2.2. **Renormalization.** Next we want to determine when our Coxeter type representation can be unitarized. It will be convenient to replace the vectors α_i by the vectors $\beta_i = q_i^{-1/4}\alpha_i$. Then we have

$$\langle\beta_i,\beta_i\rangle = q_i^{-1/2}+q_i^{1/2}, \quad \langle\beta_i,\beta_j\rangle = -2\mathrm{co}(q_i,q_j;\frac{\pi}{m(i,j)}) = -2\cos(\pi/m(i,j)), \tag{2.1}$$

where the last equality holds if $q_1 = q_2$. In the following we assume that the elements q_i are invertible in our ring, and that there exists an involutive algebra homomorphism $\bar{\ }$ over our ground ring such that $\bar{1} = 1$ and $\bar{q}_i = q_i^{-1}, i = 1, 2$. The most common examples for our ground ring would be the rational functions in variables q_i, with suitable algebraic functions adjoined, or the complex numbers with q_i being numbers of absolute value 1. The notions of conjugate linear maps and sesquilinear forms extend to our slightly more general setting in the obvious way.

Proposition 2.2. *The pairings in Eq. 2.1 extend to a unique invariant sesquilinear form $\langle\ ,\ \rangle_S$ on V; here invariant means that $\langle T_i v, T_i w\rangle_S = \langle v, w\rangle_S$ for $v, w \in V$ and $i = 1, 2, \ldots n$.*

Proof. It follows directly from Eq 1.1 by induction on n that χ_n is a homogeneous polynomial of degree n in the variables q_1, q_2 and $x = \chi_1$, if we define $\deg(x) = 1$ and $\deg(q_i) = 2$, for $i = 1, 2$. Hence any solution x of $\chi_m = 0$ is a homogeneous algebraic function in q_1 and q_2 of degree 1. This entails $y = 2\mathrm{co}(q_1, q_2'\pi/(m+1)) = (q_1 q_2)^{-1/4} x$ has degree 0, i.e. it is an algebraic function in $q_1 q_2^{-1}$. Hence $\bar{y}(q_1, q_2) = y(q_2, q_1) = \hat{y}$. As $\hat{\chi}_m = \chi_m$ for m odd, by Lemma 1.1, it follows $\bar{y} = y$ in this case. If m is even, we have $q_1 = q_2$ and therefore automatically $\bar{y} = y$. So the coefficients $\langle \beta_i, \beta_j\rangle$ are all fixed by the involution $\bar{\ }$. Hence we can define a sesquilinear form $\langle\ ,\ \rangle_S$ on V by

$$\langle \sum_i a_i \beta_i, \sum_j b_j \beta_j \rangle_S = \sum_{i,j} a_i \bar{b}_j \langle \beta_i, \beta_j \rangle_S.$$

If q_i is invertible, one can easily calculate from relation (H) that $1 - q_i^{-1} - q_i^{-1} T_i$ is the inverse of g_i. One now shows by essentially the same computation as in the proof of Lemma 2.1, using the basis (β_i), that $\langle T_i v, w\rangle_S = \langle v, T_i^{-1} w\rangle_S$ for $v, w \in V$ and $i = 1, 2, \ldots n$.

Remark 2.3. 1. In the following we shall primarily be interested in the sesquilinear form defined in the last proposition. We shall therefore denote it just by $\langle\ ,\ \rangle$ for simplicity of notation. Several of the following constructions, such as e.g. the Gram-Schmid procedure can be easily adapted to the corresponding bilinear form defined before.

2. If we take the Coxeter graph of type A_n, we obtain a representation of Artin's braid group which is equivalent to the famous (reduced) Burau representation. Of course, our construction yields representations of braid groups for any Coxeter graph.

2.3. **Gram-Schmid procedure.** We would like to determine for which choices of q_i the Hermitian form defined in Prop. 2.2 becomes an inner product. To do so, we simply apply the Gram-Schmid procedure to determine an orthonormal basis, if possible. We shall denote the sesquilinear form constructed in the previous Section just by $\langle\ ,\ \rangle$; the following constructions would work as well for the corresponding bilinear form.

Let us choose for the set S the numbers $1, 2, \ldots |S|$, if S is finite, or $\mathbb{N}$, if S is countable. Let us assume for the moment that the restriction of $\langle\ ,\ \rangle$ to the span V_{i-1} of the vectors $\beta_1, \ldots \beta_{i-1}$ is nondegenerate, and let E_{i-1} be the orthogonal projection onto V_{i-1}. Then the i-th vector v_i of the Gram-Schmid process is given by

$$v_i = (\beta_i - E_{i-1}\beta_i)/\|\beta_i - E_{i-1}\beta_i\|, \tag{2.2}$$

provided $\|\beta_i - E_{i-1}\beta_i\| \neq 0$. We will use these norms to define functions P_i inductively by $P_n = 0$ for $n < 0$, $P_0 = 1$ and

$$P_i = \|\beta_i - E_{i-1}\beta_i\|^2 P_{i-1} = ((q^{1/2} + q^{-1/2}) - \|E_{i-1}\beta_i\|^2)P_{i-1}. \tag{2.3}$$

Hence, if v_i is well-defined, we obtain from the last two equations

$$\langle \beta_i, v_i\rangle^2 = P_{i-1}/P_i. \tag{2.4}$$

The P_i's obviously are functions of q_1, ..., q_i. Moreover, we have $P_1 = q_1^{1/2} + q_1^{-1/2}$. We shall see that the P_i's are Laurent polynomials in the variables $q_j^{1/2}$ in many cases. The following two lemmas are useful for calculating the functions P_i.

Lemma 2.4. *Let $i \in S$. We assume that $\langle \beta_i, \beta_j\rangle = 0$ for all $j \leq i-2$. Then*
(a) $\langle \beta_i, v_{i-1}\rangle = \sqrt{P_{i-2}/P_{i-1}}\langle \beta_i, \beta_{i-1}\rangle$, and
(b) $P_i = (q_i^{1/2} + q_i^{-1/2})P_{i-1} - \langle \beta_i, \beta_{i-1}\rangle^2 P_{i-2}$.

Proof. By definition and assumptions, we have $\|\beta_j - E_{j-1}\beta_i\|^2 = P_{j-1}/P_j$ for $j = i, i-1$. If $(v_k)_k$ is the orthonormal basis obtained from the β_k via Gram-Schmid, it follows from Eq. 2.2 that $v_{i-1} = \sqrt{P_{i-1}/P_{i-2}}\beta_{i-1} + \gamma$, with $\gamma \in V_{i-2}$. As $\langle \beta_i, \gamma\rangle = 0$ by assumption, we obtain (a). As $E_{i-1}\beta_i = \sum_{j=1}^{i-1}\langle \beta_i, v_j\rangle v_j$ and as $\langle \beta_i, v_j\rangle = 0$ for $j < i-1$, we have

$$\|E_{i-1}\beta_i\|^2 = \|\langle \beta_i, v_{i-1}\rangle\|^2 = P_{i-1}/P_{i-2}\langle \beta_i, \beta_{i-1}\rangle^2.$$

It follows that

$$\|\beta_i - E_{i-1}\beta_i\|^2 = \|\beta_i\|^2 - \|E_{i-1}\beta_i\|^2 = q_i^{1/2} + q_i^{-1/2} - P_{i-1}/P_{i-2}\langle \beta_i, \beta_{i-1}\rangle^2.$$

Multiplying this equation by P_{i-2} shows (b).

Lemma 2.5. *Assume that $i-2$ is a simple triple point of our graph, i.e. it is connected only to the vertices i, $i-1$ and $i-3$, by single edges, and both i and $i-1$ are not connected to any vertex $j < i-2$. Then we have, for $q = q_i$,*

$$P_i = (q^{1/2} + q^{-1/2})(P_{i-1} - P_{i-3}).$$

Proof. Let (v_j) be the orthonormal basis obtained by Gram-Schmid for $j < i$. As $\langle \beta_i, \beta_j\rangle = 0$ for $j < i-1$, we can write $E_{i-1}(\beta_i) = x_1 v_{i-2} + x_2 v_{i-1}$. As $\langle \beta_i, \beta_j\rangle = \langle \beta_{i-1}, \beta_j\rangle$ for $j < i-1$, we have

$$x_1 = \langle \beta_{i-1}, v_{i-2}\rangle = -\sqrt{P_{i-3}/P_{i-2}},$$

by Lemma 2.4,(a). As β_{i-1} is in the span of v_{i-2} and v_{i-1} and has norm $q^{1/2} + q^{-1/2}$, we can chose v_{i-1} such that $\langle \beta_{i-1}, v_{i-1}\rangle = \sqrt{P_{i-1}/P_{i-2}}$, using Lemma 2.4,(b). Hence we derive from $\langle \beta_{i-1}, E_{i-1}(\beta_i)\rangle = 0$ that

$$-x_1\sqrt{P_{i-3}/P_{i-2}} + x_2\sqrt{P_{i-1}/P_{i-2}} = 0.$$

One calculates from this that $x_2 = P_{i-3}/\sqrt{P_{i-1}P_{i-2}}$. We can now easily compute $\|\beta_i - E_i(\beta_i)\|^2 = q^{1/2} + q^{-1/2} - x_1^2 - x_2^2$, using the identities of Lemma 2.4. The expression for P_i follows from this.

Lemma 2.6. *The sesquilinear form $\langle\ ,\ \rangle$ is an inner product on the vector space V_n if and only if $P_1, P_2, \ldots, P_n$ are positive.*

Proof. By construction, we obtain an orthogonal basis from the vectors $\beta_i - E_{i-1}\beta_i$, $i \in S$, provided the projections E_{i-1} are well-defined. The latter statement is the case if and only if the polynomials P_j are nonzero for $1 \leq j < i$. As $\|\beta_i - E_{i-1}\beta_i\|^2 = P_{i-1}/P_i$, we see that our sesquilinear form is positive definite if and only if all these square norms are positive, which is equivalent to all the functions P_i being positive.

2.4. **Polynomials for Weyl groups.** We compute the functions P_i for the Coxeter graphs corresponding to Weyl groups, using Lemmas 2.4 and 2.5. We choose the labelling for the graph B_n such that the endpoint with the double edge is labelled by 1, and with $i+1$ being the vertex not labelled yet which is connected with the vertex i. For type D_n, it will be convenient to start the labeling at the endpoint of the longest leg, with the endpoints next to the triple point being labeled by $n-1$ and n, Moreover, observe that we only have single edges for types A_n and D_n, and also for all edges of B_n except for the one between 1 and 2. So we can set $q_i = q$ for all i in A_n and D_n, and for all $i \neq 1$ in type B_n. We set $q_1 = Q$ in type B_n. Then we get

$$A_n: \quad P_i = \frac{q^{(i+1)/2} - q^{-(i+1)/2}}{q^{1/2} - q^{-1/2}} = q^{i/2} + q^{(i-2)/2} + \ \ldots\ + q^{-i/2},$$

$$B_n: \quad P_i = Q^{1/2}q^{(i-1)/2} + Q^{-1/2}q^{-(i-1)/2}.$$

For type D_n, the first $n-1$ functions P_i will coincide with the ones for A_n. Using Lemma 2.5, we obtain

$$P_n = (q^{(n-1)/2} + q^{-(n-1)/2})(q^{1/2} + q^{-1/2}) = q^{n/2} + q^{(n-2)/2} + q^{-n/2} + q^{-(n-2)/2}.$$

Using the labelling coming from the extensions of graphs $A_4 \subset D_5 \subset E_6 \subset E_7 \subset E_8$, we get for P_5 the polynomial in the last formula for $n = 5$, and, by Lemma 2.4,

$$E_6: \quad P_6 = (q^1 + 1 + q^{-1})(q^2 - 1 + q^{-2}),$$

$$E_7: \quad P_7 = (q^{1/2} + q^{-1/2})(q^3 - 1 + q^{-3}),$$

$$E_8: \quad P_8 = (q^4 + q^3 - q^1 - 1 - q^{-1} + q^{-3} + q^{-4}).$$

For G_2, we get, as usual, $P_1 = q^{1/2} + q^{-1/2}$ and

$$G_2: \quad P_2 = Q^{1/2}q^{1/2} - 1 + q^{-1/2}Q^{-1/2}.$$

Finally, we get for F_4 the polynomials

$$F_4: \quad P_1 = Q^{1/2}+Q^{-1/2}, \quad P_2 = Q+1+Q^{-1}, \quad P_3 = Q^{1/2}q^{1/2}+Q^{-1/2}q^{-1/2}, \quad P_4 = Qq-1+Q^{-1}q^{-1}.$$

Theorem 2.7. *Consider a Hecke algebra corresponding to a Weyl group, and assume $Q = q$ in the nonsimply laced cases $BCFG$. Its Coxeter type representation is unitary if and only if $q = e^{2\pi it}$, with $|t| \leq 1/h$, where h is the Coxeter number.*

Proof. One observes that for $Q = q$ the zeros of all the polynomials P_i computed in the previous section are roots of unity. It is not hard to check that the highest degree of such roots of unity coincides with the Coxeter number of the graph (e.g. for E_8 it would be 30).

Remark 2.8. If we set $Q = e^{2\pi is}$, we can also easily express the values for which our Coxeter type representation is unitary. This is left as an exercise to the interested reader.

2.5. **Affine Hecke algebras.** We briefly look at Hecke algebras corresponding to affine reflection groups. We shall show for affine type A in some detail why our Coxeter representation can never be unitarized, except for $q = 1$ when our sesquilinear form becomes positive semidefinite. Recall that the Coxeter graph for affine type $\hat{A}_n$ can be described as the boundary of a polygon with $n + 1$ sides.

Lemma 2.9. *The polynomials P_i for affine type $\hat{A}_n$ are equal to the ones for A_n if $i \leq n$, and $P_{n+1} = (q^{(n+1)/4} - q^{-(n+1)/4})^2$.*

Proof. This is a straightforward calculation. We give some details for the interested reader. Let (v_j) be the Gram-Schmid orthonormal basis for type A_n. Then we can write vectors β_i by $\beta_1 = (q^{1/2} + q^{-1/2})v_1$ and

$$\beta_i = -\sqrt{P_{i-2}/P_{i-1}}v_{i-1} + \sqrt{P_i/P_{i-1}}v_i.$$

for $2 \leq i \leq n$. Let $x_i = \langle \beta_{n+1}, v_i \rangle$. Then $\langle \beta_{n+1}, v_1 \rangle = -1$ implies $x_1 = -\sqrt{P_0/P_1}$, and one shows by induction on i, using $\langle \beta_{n+1}, \beta_i \rangle = 0$, that $x_i = -1/\sqrt{P_{i-1}P_i}$ for $2 \leq i < n$. Similarly, one calculates from $\langle \beta_{n+1}, \beta_n \rangle = -1$ that $x_n = -(P_{n-1} + 1)/\sqrt{P_{n-1}P_n}$.

It is easy to show by induction on i that $\sum_{j=1}^{i} x_j^2 = P_{i-1}/P_i$ for $i < n$, and $\|E_n(\beta_{n+1})\|^2 = \sum_{j=1}^{n} x_j^2 = 2(P_{n-1} + 1)/P_n$. It follows that

$$P_{n+1} = (q^{1/2} + q^{-1/2})P_n - 2P_{n-1} - 2 = q^{(n+1)/2} - 2 + q^{-(n+1)/2}.$$

Corollary 2.10. *The Coxeter type representation for affine Hecke algebras of type $\hat{A}_n$ can not be unitarized.*

Proof. If $q = e^{2\pi it}$, we have $P_{n+1}(q) = -4\sin^2(n + 1)t\pi/2$.

Remark 2.11. Similar statements can be shown for other affine Hecke algebras. Here the graphs have no cycles, and the additional polynomial can be calculated fairly easily, using Lemmas 2.4 and 2.5.

3. Some Applications

3.1. **Support projections.** Let X be a graph with n edges, and let $k < n$. Let us assume that the bilinear form $\langle \ , \ \rangle$ defined in the previous section is nondegenerate on V as well as on V_k, the span of $\{\beta_i, 1 \leq i \leq k\}$. We define f_k to be the projection onto the orthogonal complement of V_k. Moreover, we also define the element $e_i \in \mathrm{End}(V)$ by

$$e_i : v \in V \mapsto e_i(v) = \langle v, \beta_i \rangle \beta_i.$$

Observe that e_i is a scalar multiple of the projection onto β_i, with the multiple being $\|\beta_i\|^2$. Then we have the following easy

Lemma 3.1. *The idempotents f_k are defined inductively by $f_0 = 1$, and by*

$$f_k = f_{k-1} - \frac{P_{k-1}}{P_k} f_{k-1} e_k f_{k-1}, \tag{3.1}$$

where e_i is the orthogonal projection onto the span of β_i.

Proof. As the image of e_k is contained in the span of $\{v_j, j \leq k\}$, the element $f_{k-1}e_k f_{k-1}$ must be a multiple of the projection onto v_k. This multiple is equal to

$$\langle v_k, f_{k-1} e_k f_{k-1} v_k \rangle = \langle v_k, e_k v_k \rangle = \langle \beta_k, v_k \rangle^2.$$

The claim now follows from Eq. 2.4.

Remark 3.2. For Coxeter graph A_n, one can use the formula in the previous lemma to define elements f_k inductively, where we take for e_i the element in the corresponding Hecke algebra defined by $e_i = -q^{-1/2}(g_i - q)$. Observe that the braid relation for type A, i.e. with only single edges, can be expressed equivalently by

$$e_i e_{i+1} e_i - \frac{q}{(1+q)^2} e_i = e_{i+1} e_i e_{i+1} - \frac{q}{(1+q)^2} e_{i+1}.$$

It is then a fairly straightforward proof by induction to show that the f_ks are central idempotents in the Hecke algebra uniquely determined by $e_i f_k = 0 = f_k e_i$ for all $i \leq k$. This has been important for simplifying Jones' proof for restriction of index values of subfactors (see [Jo], [W1]). The formula has also found applications in mathematical physics and low-dimensional topology, see e.g. [FRS], [KL], [Li], [MV].

Unfortunately, it seems that such a widespread application of the formula in Lemma 3.1 does not seem to hold for other graphs.

3.2. **Basics about Hecke algebras.** We can now use our results to derive some basic results about Hecke algebras in a fairly easy way. We need some terminology from the theory of reflection groups; see [B], [H] for more details. Let X be a graph, and let $W = W(X)$ be the corresponding reflection group; here the generators, denoted by s_i, satisfy besides the braid relations also $s_i^2 = 1$ for all i. We speak of a *reduced expression* of an element $w \in W$ if it is written as a product of generators with the minimum number of factors; that number is called the *length* of w, denoted by $\ell(w)$. It is known that the element $T_w = T_{s_1} T_{s_2} \ldots T_{s_r}$, where $s_1 s_2 \ldots s_r$ is a reduced expression for w, is well-defined independent of the choice of the reduced expression. It is quite easy to check that the elements T_w, with $w \in W$, span the Hecke algebra $H = H(X)$. We want to prove that they are also linearly independent.

We will basically follow the standard approach, which goes as follows. Assume linear independence. Define the vector space V with basis $\{v_w, w \in W(X)\}$. Identifying the vectors v_w with the elements T_w, it follows from the multiplicative structure of the Hecke algebra that

$$T_i v_w = \begin{cases} v_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ (q_i - 1)v_w + q_i v_{s_i w} & \text{if } \ell(s_w) < \ell(w). \end{cases} \tag{3.2}$$

On the other hand, if we can show that the maps defined above do indeed define a representation of the Hecke algebra, then the T_w are linearly independent; indeed, already the elements $T_w v_1 = v_w$ are linearly independent. See [B], [H] for more details.

3.3. **Rank 2 Case.** A Hecke algebra is called of rank 2 if the corresponding graph has exactly two vertices. Hence the only variation is given by the number of edges between these two vertices. We also include the case of infinitely many vertices, in which case the braid relation becomes vacuous.

Lemma 3.3. *A Hecke algebra of rank 2 has a basis labeled by the elements of the corresponding Coxeter group.*

Proof. . If we have $0 < m - 2 < \infty$ edges, the corresponding reflection group is the dihedral group of order $2m$. It follows from Eq. 1.6 that we obtain for any integer $0 < j \leq m$ a representation with respect to the basis $\{\alpha_1, \alpha_2\}$ given by the matrices

$$T_1 \quad \mapsto \quad \begin{pmatrix} -1 & -\chi_1 \\ 0 & q_1 \end{pmatrix} \quad \text{and} \quad T_2 \quad \mapsto \quad \begin{pmatrix} q_2 & 0 \\ -\chi_1 & -1 \end{pmatrix}, \tag{3.3}$$

where $\chi_1 = (q_1 q_2)^{1/4} 2\text{co}(q_1, q_2; j\pi/m)$. It is now easy to check that the trace of the matrix representing $T_1 T_2$ is equal to $4(q_1 q_2)^{1/2}\text{co}^2(j\pi/m) - q_1 - q_2$. Hence these representations are mutually nonisomorphic for $0 < j < m/2$. Moreover, if m is even, we have four mutually nonisomorphic one-dimensional representations, while for m odd, we have two nonisomorphic one-dimensional representations. Forming the direct sum of all of these representations, we obtain a representation of the Hecke algebra whose image has dimension $2m$. This is the order of the corresponding dihedral Coxeter group, and hence the elements T_w labeled by the elements $w \in W$ are linearly independent.

If we have infinitely many edges between the two vertices, there is nothing to show.

3.4. **General case.** Let $W_I \subset W$ be a subgroup of W generated by a subset of the generators s_i. Then it is well-known that there exists for each coset of W_I in W an element w_0 such that

$$\ell(w w_o) = \ell(w_0) + \ell(w) \tag{3.4}$$

for any $w \in W_I$. This can be shown by picking an element w_0 of minimal length in the given coset. If the additivity property did not hold, one could find an element of shorter length in the coset, using the deletion condition (see e.g. [H] Section 5.8).

Proposition 3.4. *The elements T_w, $w \in W(X)$ form a basis for the Hecke algebra $H(X)$.*

Proof. We follow the standard proof, i.e. we have to show that the actions of the elements T_i on the vector space V defined in Eq. 3.2 define a representation of the Hecke algebra. Observe that each relation only involves two elements, say T_i and T_j. Let $W(i,j)$ be the reflection group generated by s_i and s_j, and let $H(i,j)$ be the corresponding Hecke algebra. Then for each $w_0 \in W(X)$, the span V_{w_0} of $\{v_{ww_0}, w \in W(i,j)\}$ is invariant under both T_i and T_j. Let $V(i,j)$ be the span of vectors v_w, $w \in W(i,j)$. By Lemma 3.3 and the discussion in Section 3.2 we obtain a representation of $H(i,j)$ on $V(i,j)$. If we pick for w_0 the element of minimum length in our given coset, it follows from Eq 3.2 and 3.4 that the map $v_w \in V(i,j) \mapsto v_{ww_0}$, commutes with the action of $H(i,j)$. Hence the commutation relations between T_i and T_j also hold on V_{w_0}.

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