EXERCISES MATH 202B - 3rd Assignment

1. Let V be a vector space, and let $W = V^m = V \oplus V \oplus \dots \oplus V$ (m summands). Moreover, let $\psi_j : v \in V \mapsto (0, 0, \dots, v, \dots 0)$ (v in the j-th slot) and let $\pi_i : (v_j)_j \in V^m \mapsto v_i \in V$. Show:
(a) Let $b \in \mathrm{End}(W)$ and let $b_{ij} = \pi_i \circ b \circ \psi_j$. Show that the map $b \in \mathrm{End}(W) \mapsto (b_{ij}) \in M_m(\mathrm{End}(V))$ is an isomorphism of algebras.
(b) Use (a) to show that $\mathrm{End}_A(W) \cong M_m(\mathrm{End}_A(V))$, with notations as in (a) for the A-modules V and W, where A is an algebra.

2. Let $\mathcal{A}$ be the algebra of $n \times n$ matrices over some field F and let A be an $n \times n$ matrix, and let $W = \mathcal{A}A$ be the $\mathcal{A}$-module (or $\mathcal{A}$-left ideal) generated by A.
(a) What are the possible dimensions for W? Give an example of A for each such dimension.
(b) Determine the dimension of W in terms of some numerical invariant associated to a matrix A (such as e.g. the trace, determinant, nullity etc). Obviously not all and possibly none of the given examples may be correct.

3. Let $A = \mathbf{C}\mathbf{Z}/3$ and let W be an A-module such that the matrices representing the group action all have real coefficients. Determine $\mathrm{End}_A W$ if $Tr(\bar{0}) = 8$ and $Tr(\bar{1}) = -1$.

4. Let A be the algebra of all upper triangular 2×2 matrices over a field F. Consider the A-module $V = F^2$, with the usual module action given by matrix multiplication. Compute $B = \mathrm{End}_A(V)$ and $\mathrm{End}_B(V)$. Why is your result compatible with the von Neumann-Jacobson density theorem?

5. Let V, W be vector spaces, and let $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$. Define $i(a, b) \in \mathrm{End}(V \otimes W)$ by $i(a, b)(v \otimes w) = a(v) \otimes b(w)$
(a) Check that indeed $i(a, b)$ is a linear map.
(b) Show that the map $(a, b) \in \mathrm{End}(V) \times \mathrm{End}(W) \to i(a, b) \in \mathrm{End}(V \otimes W)$ is bilinear.
(c) Show that the map i above induces an isomorphism between $\mathrm{End}(V) \otimes \mathrm{End}(W)$ and $\mathrm{End}(V \otimes W)$.