EXERCISES MATH 202B - 7th Assignment

1. Show that all simple characters of S_n are integer valued, for all $n \in \mathbf{N}$.
2. Let A be a semisimple algebra over $\mathbf{C}$, and let $p \in A$ be an idempotent. Let $Ap = \bigoplus m_\lambda V_\lambda$ be the decomposition of Ap into a direct sum of irreducible A-modules. Show that $m_\lambda = \chi_\lambda(p)$, where χ_λ is the trace on $\mathrm{End}(V_\lambda)$.
3. Let ε be the sign character of S_n, and let χ_λ be the irreducible character of S_n corresponding to the Young diagram λ. Show that $\varepsilon\chi_\lambda = \chi_{\lambda^T}$, where λ^T is the Young diagram with λ_i boxes in its i-th *column* for $i = 1, 2\ \ldots$. *Hint*: As $\varepsilon\chi_\lambda$ is simple (why?), it would suffice to show that $(\varepsilon\chi_\lambda)(\frac{1}{\alpha_{\lambda^T}} q_{\lambda^T} p_{\lambda^T}) = 1$, where α_{λ^T} is the scalar such that $\frac{1}{\alpha_{\lambda^T}} q_{\lambda^T} p_{\lambda^T}$ is an idempotent (why?).
4. (a) Show that $\dim V_\lambda = \dim V_{\lambda^T}$ for any simple S_n-module V_λ.

(b) Show that V_λ and V_{λ^T} are isomorphic A_n-modules.

(c) Let W_λ be the S_n module induced from the A_n module V_λ as above. Prove that $W_\lambda \cong V_\lambda \oplus V_{\lambda^T}$ as an S_n-module, if $\lambda \neq \lambda^T$.

(d) Show that V_λ remains irreducible as an A_n-module if $\lambda \neq \lambda^T$. (Hint: Use Frobenius reciprocity).