EXERCISES MATH 202B - 8th Assignment

One of the main theorems of our course will be to show that $\tilde{s}_\lambda = s_\lambda$ (recall that $\tilde{s}_\lambda$ was defined as a trace of a certain element, and s_λ was defined as a quotient of determinants). You can assume this result (which we will prove) in the following. Also recall that $\Delta = \prod_{i<j}(x_i - x_j)$.

1. (a) Let P be a symmetric polynomial. It was shown in class that we can write $P = \sum_\lambda \alpha_\lambda s_\lambda$. Show that α_λ is the coefficient of $x^{\lambda+\rho}$ in the polynomial ΔP; here $\rho = (k-1, k-2, \ ... \ 1, 0)$.

   (b) Show that the character $\chi_\lambda(\pi)$ of a permutation π with cycle structure $C = (c_i)$, where c_i is the number of i-cycles, is equal to to the coefficient of $x^{\lambda+\rho}$ in the polynomial $\Delta \prod_{i=1}^k (x_1^i + \ ... \ + x_k^i)^{c_i}$. This is Frobenius' formula.

2. Use Frobenius' formula to calculate $\chi_\lambda(\pi)$ in terms of its cycle structure $C = (c_i)$ (as in 1) for $\lambda = [n-1, 1]$ and for $\lambda = [n-2, 2]$. E.g. the result is equal to $c_1 - 1$ for $\lambda = [n-1, 1]$. What about $\lambda = [n-3, 3]$?