

EXERCISES MATH 202C - 5th Assignment

1. (a) Let I be a monomial ideal in $k[x_1, \dots, x_n]$. Let $S_1 = \{x^{\alpha(i)}, 1 \leq i \leq r\}$ and $S_2 = \{x^{\beta(j)}, 1 \leq j \leq s\}$ be two minimal generating sets (i.e. if you remove one element from one of these sets, the ideal generated by the smaller set will be smaller). Show that $S_1 = S_2$.
 (b) Show that $x^\beta \in I$ if and only if $x^{\alpha(i)} | x^\beta$ for some $\alpha(i)$ as in (a).
 (c) Prove or disprove: There exists a number M such that any monomial ideal in $k[x, y]$ can be presented with $\leq M$ generators.
2. (a) Show that $S(x^\alpha f, x^\beta g) = x^\gamma S(f, g)$ and calculate γ . Moreover, show that the degree of $S(x^\alpha f, x^\beta g)$ is smaller than the degree of $LCM(LT(x^\alpha f), LT(x^\beta g))$.
 (b) Let f, g_1, \dots, g_s be polynomials in $k[x_1, \dots, x_n]$. Do division with remainder of f by the polynomials g_i to get $f = \sum a_i g_i + r$. Show that $\max\{\deg(a_i f_i), \deg(r)\} = \deg(f)$.
3. (a) Calculate a Gröbner basis for the ideal $I = \langle x^2 y - 1, xy^2 - x \rangle$.
 (b) Calculate a reduced Gröbner basis for the ideal I in (a).
4. (a) Let $I \subset k[x_1, \dots, x_n]$ be an ideal and let $g_1, g_2, \dots, g_s \in I$. Show that if $\langle LT(g_i), 1 \leq i \leq s \rangle = LT(I)$, then $\langle g_i, 1 \leq i \leq s \rangle = I$.
 (b) Compare the number of elements of a reduced Gröbner basis for I with the number of elements in a minimum set of generators of I .