EXERCISES MATH 202C - Second Assignment

1. Let λ be a Young diagram with N boxes and let $l_i = \lambda_i + N - i$. Show that the coefficient of $x_1 x_2 \ldots x_N$ of $s_\lambda(x_1, \ldots, x_N)$ is equal to

$$\frac{N!}{l_1! l_2! \ldots l_N!} \prod_{i<j} (l_i - l_j).$$

2. Let λ be a Young diagram and let N be an integer $\geq$ the number of rows of λ. Let $l_i = \lambda_i + N - i$. Finish the proof of Lemma 8.13, which states that

$$\prod_{i<j} (l_i - l_j) \prod_{(r,s)\in\lambda} h(r,s) = l_1! l_2! \ldots l_N!.$$

   Recall that we have already shown the claim if N is equal to the number of rows and if $\lambda_N > 1$.
   (a) Show that if the formula holds for N being equal the number of rows of λ, it also holds for any larger N.
   (b) Show the claim with $\lambda_N = 1$ and N being equal the number of rows of λ. Again, consider the diagram μ obtained by removing the first column of λ.

3. For this problem you may assume that $e_\lambda V^{\otimes n}$ is a simple $Gl(N)$ module whenever e_λ is a minimal idempotent in $(\mathbf{C}S_n)_\lambda$ and λ is a Young diagram with $\leq N$ rows.
   (a) Let $\dim V = 3$. Describe the decomposition of $V^{\otimes 4}$ into irreducible $Gl(3)$-modules and into irreducible S_4-modules (with multiplicities).
   (b) Let $W_\lambda = e_\lambda V^{\otimes n}$, where e_λ is a minimal idempotent in $\mathbf{C}S_n$ corresponding to a Young diagram λ with n boxes, and $\dim V > 10$. Calculate the decomposition of the $Gl(V)$-module $W_{[5,3,3,2]} \otimes W_{[1,1,1]}$ into a direct sum of irreducible $Gl(V)$-modules.