EXERCISES MATH 202C - Seventh Assignment

1. This exercise fills in some of the 'standard verifications' for Lemma 12.4. As usual, let $g \in G \mapsto A_g$ be a representation of the finite group G into the $n \times n$ matrices, acting on the vector space $V = \mathbf{C}^n$. Moreover, let B be an $n \times n$ matrix.
(a) Let $a_B(f)(x_1, \ ..., \ x_n) = f((x_1, \ ..., \ x_n)B)$. Show that a_B is an automorphism of $k[x_1, \ ..., \ x_n]$ which preserves the gradation. This means $a_B(fg) = a_B(f)a_B(g)$, $a_B(f+g) = a_B(f)+a_B(g)$ and a_B maps $k[x_1, \ ..., \ x_n]_d$ into itself, where $k[x_1, \ ..., \ x_n]_d$ is the span of all monomials of degree d.
(b) Define, as usual, $(g \cdot f)(x_1, \ ..., \ x_n) = f((x_1, \ ..., \ x_n)A_g)$, and let $k[x_1, \ ..., \ x_n]^G$ be the subalgebra of G-invariant polynomials. How does $k[x_1, \ ..., \ x_n]^G$ change if we replace the given representation of G by $g \mapsto BA_gB^{-1}$, for a fixed invertible matrix B?
(c) Calculate the matrix which describes the action of $g \in G$ on $k[x_1, \ ..., \ x_n]_1$ with respect to the basis $\{x_1, \ ..., \ x_n\}$.
(d) Let $p = \frac{1}{d!}\sum_{\sigma \in S_d} \sigma$. Show that the representation of G on $k[x_1, \ ..., \ x_n]_d$ is equivalent to the representation of G on $pV^{\otimes d}$. (Hint: Consider the vectors $pv^{\otimes \alpha}$, where $v^{\otimes \alpha} = v_1^{\otimes \alpha_1} \otimes \ ... \ \otimes v_n^{\otimes \alpha_n}$.)
(e) Let χ_d be the character of the representation of G on $k[x_1, \ ..., \ x_n]_d$. Show that χ_d is independent under basis transformations for V. Express χ_d in terms of Schur functions and eigenvalues of A_g.

2. Let V be the 3-dimensional irreducible representation of the subgroup A_4 of even permutations in S_4.
(a) Calculate the Molien series $k[x_1, x_2, x_3]^{A_4}$.
(b) Find the degrees of a maximal set of algebraically independent elements in the ring $k[x_1, x_2, x_3]^{A_4}$, where the elements are supposed to have as small degrees as possible.
(c) What is the minimal number of generators for $k[x_1, x_2, x_3]^{A_4}$? What are the degrees of the generators added to the algebraically independent generators in (b).

3. Let $G = \mathbf{Z}/3$, acting on $k[x_1, x_2, x_3]$ via shifting the indices mod 3 (e.g. $\bar{1}.x_i = x_{i+1}$, indices mod 3).
(a) Calculate generators of $k[x_1, x_2, x_3]^G$ and check your result via Molien's theorem.
(b) Give an explicit decomposition of $k[x_1, x_2, x_3]^G$ as free module over an *hsop* (homogeneous system of parameters).
(c) Give a presentation of $k[x_1, x_2, x_3]^G$ via generators and relations.
(d) Answer question (c) for the action of $G = \mathbf{Z}/3$ on $k[y_0, y_1, y_2]$ given by $\bar{1}.y_j = \theta^j y_j$, where $\theta = e^{2\pi i/3}$.