EXERCISES MATH 202C - Eighth Assignment

1. Let $f \in k[x_1, x_2, \ ..., \ x_n]^{A_n}$, where $A_n \subset S_n$ acts, as usual, via permutations on the variables.
   (a) Show that f can be uniquely written as $f = f_1 + f_2$, where f_1 is a symmetric function, and f_2 is an antisymmetric function.
   (b) Calculate the Hilbert series of $k[x_1, x_2, \ ..., \ x_n]^{A_n}$ by explicitly finding generators and its Hironaka decomoposition (*Hint*: What is the smallest degree of an antisymmetric function in n variables?)

2. (a) Express the Hilbert series of $k[x_1, \ ..., \ x_n, x_{n+1}]^G$ in terms of the Hilbert series of $k[x_1, \ ..., \ x_n]^G$, if G acts trivially on x_{n+1}.
   (b) Do parts 2(b) and 2(c) of the previous assignment. It is not necessary to explicitly write down the generating polynomials.

3. Let B be the k algebra $k[x_1, x_2]/\langle x_1^2 x_2, x_1 x_2^2\rangle$. Show that B is *not* Cohen-Macaulay. Calculate its Hilbert series (with $\bar{x}_1$ and $\bar{x}_2$ having degree 1).

4. (a) Let $k[x_1, x_2, x_3]^{\mathbf{Z}/4}$ be the invariant ring studied in class on Friday (or see pages 42 and 43 in the book by Sturmfels). Show that $\eta_i \eta_j \in \sum_{m=1}^{4} \eta_m k[\theta_1, \theta_2, \theta_3]$ for any $1 \leq i, j \leq 4$.
   (b) Let $\mathbf{Z}/4$ act on $k[x_1, x_2, x_3, x_4]$ via shift of the index of variables by 1, i.e. $\bar{1} \cdot x_i = x_{i+1}$ mod 4. Describe $k[x_1, x_2, x_3, x_4]^{\mathbf{Z}/4}$ via generators and relations. (Hint: With all the previous results in this homework and in class, you should be able to do this without any calculations).