APPLIED ALGEBRA QUALIFYING EXAM – SPRING 2009

This part of the Applied Algebra exam will be scaled to make up 60% of the whole exam. The problems have the same value, except for the last one which will count more. Try to do as many problems as possible.

1. Let $p_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ and let d be the diagonal matrix with diagonal entries x_1, ... x_N, where $V = \mathbf{C}^N$. The matrix d acts on each factor of $V^{\otimes n}$, thereby defining a linear action on $V^{\otimes n}$. The action of S_n on $V^{\otimes n}$ is given via permuting the tensor factors.
(a) Calculate $Tr_{V^{\otimes n}}(p_n d)$.
(b) The value of $Tr_{V^{\otimes 10}}((p_4\otimes p_4\otimes p_2)d)$ can be written as a linear combination of Schur functions. Calculate the coefficient of $s_{[6,3,1]}$.
(c) Calculate the multiplicity of the simple S_{10} module $S^{[3,3,3,1]}$ in $V^{\otimes 10}$ for $\dim(V) = N = 5$ and for $N = 3$.

2. Let e_r be the r-th elementary symmetric function in the variables x_1, x_2, ..., x_n.
(a) Show that the determinant of $(\partial e_i/\partial x_j)_{1\leq i,j\leq n}$ is a homogeneous polynomial and calculate its degree.
(b) Calculate the determinant.

3. Let $\rho : G \to Gl(V)$ be a representation of the finite group G into the group $Gl(V)$ of invertible linear maps on the vector space V.
(a) Show that also the map $\hat{\rho} : g \in G \mapsto \rho(g^{-1})^t$ defines a representation, where ^{t} means the transpose of a matrix.
(b) Let χ_ρ and $\chi_{\hat{\rho}}$ be the characters of ρ and $\hat{\rho}$. Show that $\chi_{\hat{\rho}}(g) = \bar{\chi}_\rho(g)$ (i.e. the complex conjugate) for all $g \in G$.
(c) Let V be a simple G-module. Show: If W is a simple G module such that the trivial representation occurs in $V \otimes W$, then W must be isomorphic to the representation defined in (a).

4. Let $f_1 = x^2y^2 - x$ and $f_2 = x^3y - 1$.
(a) Calculate a Gröbner basis for $\langle f_1, f_2\rangle \cap k[x]$ and for $\langle f_1, f_2\rangle$, where $\langle f_1, f_2\rangle$ is the ideal in $k[x, y]$ generated by f_1 and f_2.
(b) What is the variety $V(f_1, f_2) = \{(a, b),\ f(a, b) = 0, f \in \langle f_1, f_2\rangle\}$ for $k = \mathbf{C}$?

5. Let G be the subgroup of S_4 generated by the permutations (12) and (34), and let V be the simple representation of S_4 labeled by the Young diagram $[2, 1, 1]$.
(a) Calculate the Molien series of $k[x_1, x_2, x_3]^G$.
(b) Let $\tilde{G} \cong \mathbf{Z}/2\times\mathbf{Z}/2$ with generators g_1 and g_2, and let W be the three-dimensional $\tilde{G}$ module with basis w_1, w_2, w_3 such that the action of G is given by

$$g_1 w_i = \begin{cases} -w_i & \text{if i=1,2,} \\ w_3 & \text{if i=3,} \end{cases} \qquad g_2 w_i = \begin{cases} w_1 & \text{if i=1,} \\ -w_i & \text{if i=2,3.} \end{cases}$$

Write down a Hironaka decomposition for $k[y_1, y_2, y_3]^{\tilde{G}}$.
(c) Find presentations of the rings $k[y_1, y_2, y_3]^{\tilde{G}}$ and $k[x_1, x_2, x_3]^G$ via generators and relations.