

MATH 251 LIE GROUPS

EXPONENTIAL MAPS AND  $C^\infty$  MANIFOLDS

The following remarks and exercises are meant to review some facts about  $C^\infty$  manifolds and about exponential maps. They are particularly recommended to those who would like to get a better feel about how to work with manifolds. If you are a little stuck, please ask.

1. The following exercise is supposed to give you an idea how to produce matrix  $C^\infty$  functions via power series. Let  $f(z) = \sum a_n z^n$  be a power series which converges for  $|z| < R$ , and let  $|f|(z) = \sum |a_n| z^n$  (which also converges for  $|z| < R$ ). Moreover, let  $X = (x_{ij})$  be an  $n \times n$  matrix, and let  $E_{ij}$  be the matrix unit, i.e. a matrix all of whose entries are zero except in the  $i$ -th row and  $j$ -th column, where it is 1.
  - (a) Calculate for fixed  $i, j, r, s$  the partial derivatives  $\frac{\partial}{\partial x_{ij}} X^m$  and  $\frac{\partial^2}{\partial x_{ij} \partial x_{rs}} X^m$ .
  - (b) Show that  $\left\| \frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \dots \partial x_{i_r j_r}} X^m \right\| \leq m(m-1)\dots(m-r+1) \|X\|^{m-r}$ .
  - (c) Show that  $\frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \dots \partial x_{i_r j_r}} f(X) = \sum_{m=r}^\infty \frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \dots \partial x_{i_r j_r}} X^m$ , where the right hand side converges absolutely for  $\|X\| < R$ .

2. (a) Let  $X = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ , and let  $\lambda = \sqrt{x^2 + yz} = \sqrt{-\det(X)}$ . Show that  $\exp(X) = \cosh(\lambda)I + \lambda^{-1} \sinh(\lambda)X$ ; rewrite this in terms of real functions if  $x^2 + yz < 0$ . Observe that this gives a description of a neighborhood of 1 in  $SL(2, \mathbf{R})$  in terms of the coordinates  $(x, y, z)$ . (Why is there no problem for  $\sqrt{x^2 + yz}$  at  $x^2 + yz = 0$  as far as differentiability goes?)
  - (b) Show that the matrix  $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$  is in  $SL(2, \mathbf{R})$ , but not in the image of its Lie algebra under the exponential map.
  - (c) Show that for every invertible matrix  $A$  there exists a matrix  $X$  such that  $\exp(X) = A$ . Why does this not contradict (b)?
  - (d) Show that whenever  $\|X\| < \ln 2$ , then  $\exp(X)$  is in the domain of the matrix logarithm, i.e.  $\|\exp X - I\| < 1$ .

3. Here are some indications how to explicitly check the  $C^\infty$  structure of  $SL(2, \mathbf{R})$ . It follows from Problem 1 that we have the map

$$(x, y, z) \mapsto \exp X = (a_{ij}(x, y, z))_{1 \leq i, j \leq 2},$$

where the  $a_{ij}$  are  $C^\infty$  functions, and the map is invertible if we restrict  $\exp$  to a sufficiently small neighborhood of 0 (see Problem 2(d)).

- (a) Show that the matrix coefficients of  $(x, y, z) \mapsto \exp(X)^{-1}$  are  $C^\infty$  maps.
- (b) Show that the matrix coefficients of

$$\Phi : (x_1, y_1, z_1, x_2, y_2, z_2) \mapsto \exp(X_1) \exp(X_2)$$

are  $C^\infty$  maps.

(c) Show that for  $X_1, X_2$  in sufficiently small neighborhoods  $U_1, U_2$  of 0 in the Lie algebra of  $SL_2$  the map  $\log \circ \Phi$  is a  $C^\infty$  map (this is a special case of what is mentioned in Problem 4).

(d) Explain what remains to be done to prove that  $SL(2, \mathbf{R})$  is a Lie group.

4. By definition, a map  $f$  from an open set  $V \subset \mathbf{R}^m$  to a  $d$ -dimensional Lie group (or any  $C^\infty$ -manifold) is a  $C^\infty$  map if the maps  $\phi_\lambda \circ f : V \rightarrow \mathbf{R}^d$  are  $C^\infty$  maps for all coordinate maps  $\phi_\lambda : U_\lambda \rightarrow \mathbf{R}^d$ . If  $G$  is a matrix Lie group, one can show that this is equivalent to the fact that all matrix coefficients of  $f$  are  $C^\infty$  functions. Check this for the example of  $G = SL(2, \mathbf{R})$ , where you can assume that the image of  $f$  is contained in the domain of  $\log$ , i.e. in the set  $\{A, \|A - I\| < 1\}$ .
5. Let  $H \subset G$  be matrix Lie groups. Show that the Lie algebra  $\mathfrak{h}$  of  $H$  is a sub Lie algebra of the Lie algebra  $\mathfrak{g}$  of  $G$ , i.e. it is a linear subspace  $\mathfrak{h} \subset \mathfrak{g}$  such that  $[X, Y] \in \mathfrak{h}$  if  $X, Y \in \mathfrak{h}$ .
6. *Remarks.* For practical calculations one usually does not use the exponential maps for charts. Many examples are given via algebraic equations (such as determinant = 1, or  $A^T A = I$  (orthogonality),  $U^* U = I$  (unitarity) etc). The existence of charts then follows from the implicit function theorem (or, in simpler cases, by just solving for sufficiently many matrix entries using the defining equations).

Answer: 1(a)  $\sum_{p=1}^m X^{p-1} E_{ij} X^{m-p}$  and the sum of all possible expressions obtained by replacing two of the factors in  $X^m$  by  $E_{ij}$  and  $E_{rs}$  (check there are  $m(m-1)$  such expressions).