Math 102
Winter '08
Homework #3

2.2

(5) (A) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$

$\Rightarrow u+2v+2w=1 \Rightarrow u+2v+4=1 \Rightarrow u=-2v-3$
$w=2$

$\Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2v-3 \\ v \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + v\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$

(B) $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow \boxed{\text{no solution}}$

(8) $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2-2b_1 \end{array}\right)$

consistent for all choices of b_1 and b_2.

(11) "x_n" just means "a vector in the Nullspace of A"

$A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $b=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $Ax=b$ has no solution,

but A <u>does</u> have a nullspace:

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1+x_2=0 \Rightarrow x_1=-x_2$

$\Rightarrow \text{Nul } A = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

So take $x_N = \begin{pmatrix} -12 \\ 12 \end{pmatrix}$, or any element of $\text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

24 Claim: $\mathrm{rank}(AB) \le \mathrm{rank}(B)$.

Proof: Let A be $m \times n$, B be $n \times k$ $\Rightarrow$ AB is $m \times k$.

Recall the Rank + Nullity Thm:

$$\dim(\mathrm{Nul}\, AB) + \mathrm{rk}(AB) = k \qquad (*)$$
$$\dim(\mathrm{Nul}\, B) + \mathrm{rk}(B) = k.$$

Let $x \in \mathbb{R}^k$ be an element of $\mathrm{Nul}\, B$

$\Rightarrow Bx = 0$ (dfn of "nullspace")

$\Rightarrow ABx = 0$

$\Rightarrow x \in \mathrm{Nul}\, AB$.

$\therefore \mathrm{Nul}\, B \subseteq \mathrm{Nul}\, AB$.

$\Rightarrow \dim(\mathrm{Nul}\, B) \le \dim(\mathrm{Nul}\, AB)$

But then $(*)$ implies that

$\mathrm{rk}(B) \ge \mathrm{rk}(AB)$. $\boxed{\checkmark}$

25 A, B $n \times n$ such that $AB = I$.

Claim: $BA = I$.

Proof: $AB = I$, so $\mathrm{rank}(AB) = n$ (since I has full rk.)

From 24, $\mathrm{rank}(A) \ge \mathrm{rank}(AB)$

$\therefore \mathrm{rank}(A) \ge n$ (i)

But A is $n \times n$, so $\mathrm{rank}(A) \le n$ (ii)

(i) + (ii) $\Rightarrow \mathrm{rank}(A) = n$.

$\therefore A$ is invertible (square with full rank)

$\Rightarrow B = A^{-1}$

$\Rightarrow BA = I$ $\boxed{\checkmark}$

(53) (A) <u>False</u>. Let $M = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Consider $Mx = b$.

$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 = 3 \Rightarrow x_1 = 3 - x_2$

free variable

<u>Key</u>: Choose a square matrix with a nontrivial nullspace!

(B) <u>True</u>: $Ax = 0 \Rightarrow A^{-1}(Ax) = A^{-1}(0) \Rightarrow x = 0$.

So $\text{Nul}\, A = \{0\}$

$\therefore$ no free variables in any system $Ax = b$.

(C) <u>True</u> A $m \times n$

$\#$ pivot variables $\leq \min\{m, n\} \leq n$.

(D) <u>True</u>: A $m \times n$

$\#$ pivot variables $\leq \min\{m, n\} \leq m$

(70) Changing a matrix from echelon form to reduced echelon form makes the leading entry of each row equal to 1.

$\boxed{2.3}$

A

(2) $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

3 pivots $\Rightarrow \mathrm{rk}(A) = 3$

$\Rightarrow$ max # of L.I. cols $= \boxed{3}$

(9) (A)...$Ax = 0$ has a nonzero solution since A is 3×4, and therefore there is at least one free variable in the homogeneous system.

(B)...they are scalar multiples of each other ($v_1 = \alpha v_2$ for some $\alpha \in \mathbb{R}$)

(C)... $0 \cdot v_1 + \alpha \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$ for <u>all</u> choices of $\alpha \in \mathbb{R}$

$\therefore$ the homogeneous eqn. has a nontrivial solution

$\therefore$ the vectors are linearly dependent.

(12) b is in span{cols of A} when $Ax = b$ has a solution.

c is in span{rows of A} when $A^T x = c$ has a solution.

$\boxed{\text{False}}$ every vector space contains the zero vector, and Rowsp(A) is always a vector space (regardless of independence of rows).

(B) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Col A: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

pivot columns

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$ is a basis for Col A

$\Rightarrow \dim(\text{Col } A) = 2.$

Col U: has same RREF as A and ∴ same # of pivot cols, but

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$ is a basis for Col U.

$\Rightarrow \dim(\text{Col } A) = 2 = \dim(\text{Col } U).$

Row A: Row A = Col(A^T)

$A^T = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

pivot cols

∴ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$ is basis for Col(A^T) = Row(A)

$\Rightarrow \dim(\text{Row } A) = 2$

Row U: Row U = Col(U^T).

$U^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

pivot cols

∴ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ is basis for Col(U^T) = Row(U).

$\Rightarrow \dim(\text{Row } U) = 2.$

Note: $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ∴ Row U = Row A.

(20) (A) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ such that } x = y = z = w \right\}$

$= \left\{ \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} \text{ such that } x \in \mathbb{R} \right\}$

$= \left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ such that } x \in \mathbb{R} \right\}$

$\therefore S$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(B) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ such that } x + y + z + w = 0 \right\}$

$\Downarrow$

$x = -y - z - w$

$= \left\{ \begin{pmatrix} -y-z-w \\ y \\ z \\ w \end{pmatrix} \text{ such that } y, z, w \in \mathbb{R} \right\}$

$= \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\therefore S$ has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(C) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ s.t. } (x\ y\ z\ w) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0 \ \& \ (x\ y\ z\ w) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0 \right\}$.

$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ s.t. } x + y = 0,\ x + z + w = 0 \right\}$

$\Rightarrow y = z + w$

x free

z, w free.

$\longrightarrow$

20, cont'd

$$= \left\{ \begin{pmatrix} x \\ z+w \\ z \\ w \end{pmatrix} \right\} = \left\{ x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\therefore$ S has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

(D) $U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

$\uparrow \ \uparrow$

pivot cols

$\therefore$ Basis for Col $U = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= -x_4 \end{aligned}$$

So Nul $U = \left\{ \begin{pmatrix} -x_3 - x_5 \\ -x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right\} = \left\{ x_3\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$\therefore$ Basis for Nul $U = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

(21) Bases for Col U:

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\ \left\{\begin{pmatrix}\pi\\0\end{pmatrix},\begin{pmatrix}0\\37\end{pmatrix}\right\},\ \left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\6\end{pmatrix}\right\}$$

(or any set of 2 indpt vectors in $\mathbb{R}^2$).

Row U = Col(U^T)

$$U^T=\begin{pmatrix}1&0\\0&1\\1&0\\0&1\\1&0\end{pmatrix}$$

Col(U^T) has bases: $\left\{\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}\right\}$.

(26) S subspace of $\mathbb{R}^6$, $\dim S=5$.

(A) True. Let $\mathcal{B}$ = basis for S.
Then $\mathcal{B}$ is a lin indpt. set
$\therefore$ $\mathcal{B}$ can be extended to a basis for $\mathbb{R}^6$ (by 2L).

(B) False (give counterexample)
(key is that S is fixed beforehand)

(30) 2nd & 3rd cols. have pivots

$\therefore \left\{\begin{pmatrix}5\\0\\0\\0\end{pmatrix},\begin{pmatrix}4\\2\\0\\0\end{pmatrix}\right\}$ is basis for Col U.

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=0\begin{pmatrix}5\\0\\0\\0\end{pmatrix}+0\begin{pmatrix}4\\2\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}3\\1\\0\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\2\\0\\0\end{pmatrix}+\frac{1}{5}\begin{pmatrix}5\\0\\0\\0\end{pmatrix}$$