Math 102
Winter 2008
Homework #5

$\boxed{3.1}$

(2) $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}\right\}$ is a $\underbrace{\text{L.I. set}}$ that is $\underbrace{\text{not orthogonal}}$

$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ only if $c_1 = c_2 = 0$

$(1\ 1)\begin{pmatrix}2\\5\end{pmatrix} = 1\cdot 2 + 1\cdot 5 = 7 \neq 0.$

$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$ is an $\underbrace{\text{orthogonal set}}$ that is $\underbrace{\text{not L.I.}}$

$(1\ 1)\begin{pmatrix}0\\0\end{pmatrix} = 0 + 0 = 0$

$0\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ for all $c \in \mathbb{R}$!

(6) $(x\ y\ z)\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \Rightarrow x + y + z = 0$

$(x\ y\ z)\begin{pmatrix}1\\-1\\0\end{pmatrix} = 0 \Rightarrow x - y + 0 = 0$

This is what it means for $\begin{pmatrix}x\\y\\z\end{pmatrix}$ to be orthogonal to both $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

So all vectors orthog to both and must satisfy

$$\begin{matrix} x+y+z=0 \\ x=y \end{matrix} \Rightarrow \begin{matrix} z = -x-y = -2y \\ x = y \end{matrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}y\\y\\-2y\end{pmatrix} = y\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

So the subspace in question has basis $\left\{\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}$

It only has 1 vector, so it's trivially an orthogonal basis; to make it orthonormal, we need only normalize the vector:

$$\left\|\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$\therefore$ an O.N.B. for the subspace is $\boxed{\left\{\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}\right\}}$.

⑦ Note: $\text{Row } A = (\text{Nul } A)^{\perp}$ $\quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{pmatrix}$.

$\text{Col } A = (\text{Nul } A^T)^{\perp}$

(i) $x \in (\text{Row } A)^{\perp} \Rightarrow x \in \text{Nul } A$. Compute Nul A: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 3 & 6 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 + x_3 = 0 \\ x_3 = 0 \end{array}$ $\quad \therefore x = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in \text{Nul } A \Rightarrow \boxed{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in (\text{Row } A)^{\perp}}$

(ii) $y \in (\text{Col } A)^{\perp} \Rightarrow y \in \text{Nul } A^T$. Compute $\text{Nul}(A^T)$:

$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} y_1 + 2y_2 + 3y_3 = 0 \\ y_2 + y_3 = 0 \end{array}$ $\quad \therefore \boxed{y = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \in (\text{Col } A)^{\perp}}$

(iii) $z \in (\text{Nul } A)^{\perp} \Rightarrow z \in \text{Row } A$. Take $\boxed{z = (1\ 2\ 1)}$

Note THESE ANSWERS ARE NOT UNIQUE!

⑪ Claim: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, one and only one of the following systems has a solution:

(i) $Ax = b$

(ii) $A^T y = 0$ for some y s.t. $y^T b \neq 0$.

Pf: Assume, for a contradiction, that both (i) and (ii) have solutions. Then

$$0 \neq y^T b = y^T(Ax) = [y^T(Ax)]^T = (Ax)^T y = x^T \underbrace{A^T y}_{0} = 0$$

$\therefore 0 \neq 0$

This is absurd. Thus, it is not possible for both (i) and (ii) to have solutions ✓

It remains to show that either (i) or (ii) must hold. Assume (i) has no solution. Then

$b \notin \text{Col } A$

$\Rightarrow b \notin (\text{Nul } A^T)^{\perp}$

$\Rightarrow \exists\, y \in \text{Nul } A^T$ s.t. $y^T b \neq 0$

which precisely says that (ii) has a solution.

$\therefore$ Either (i) or (ii) has a solution.

Conclude that exactly one of (i), (ii) has a solution ☑

(14) Claim: $x-y$ orthogonal to $x+y \iff \|x\|=\|y\|$

Pf. ($\Rightarrow$) $x-y$ orthogonal to $x+y$

$$\Rightarrow (x-y)^T(x+y)=0$$
$$\Rightarrow (x^T-y^T)(x+y)=0$$
$$\Rightarrow x^Tx-y^Tx+x^Ty-y^Ty=0$$

But y^Tx is a scalar, so $y^Tx=(y^Tx)^T=x^Ty$

Also, $x^Tx=\|x\|^2\geq 0$ and $y^Ty=\|y\|^2\geq 0$

$\therefore \|x\|^2-\cancel{y^Tx+y^Tx}-\|y\|^2=0 \Rightarrow \|x\|^2=\|y\|^2$

$\Rightarrow \|x\|=\|y\|$ ✓

($\Leftarrow$) Assume $\|x\|=\|y\|$

WTS: $(x-y)^T(x+y)=0$

$$\begin{aligned}(x-y)^T(x+y) &= (x^T-y^T)(x+y) \quad \text{(see above)}\\ &= x^Tx-\cancel{y^Tx+x^Ty}-y^Ty\\ &= x^Tx-y^Ty\\ &= \|x\|^2-\|y\|^2\\ &= 0 \quad \text{by assumption ✓}\end{aligned}$$

☑

(16) Let $S=\left\{x\in\mathbb{R}^4 \mid x \text{ is orthogonal to both } \begin{pmatrix}1\\4\\4\\1\end{pmatrix} \text{ and } \begin{pmatrix}2\\9\\8\\2\end{pmatrix}\right\}$

Then $(x_1\ x_2\ x_3\ x_4)\begin{pmatrix}1\\4\\4\\1\end{pmatrix}=0$ and $(x_1\ x_2\ x_3\ x_4)\begin{pmatrix}2\\9\\8\\2\end{pmatrix}=0$

by definition of orthogonality, which gives us the system of equations: $x_1+4x_2+4x_3+x_4=0$

$2x_1+9x_2+8x_3+2x_4=0$

$$\Rightarrow \left(\begin{array}{cccc|c}1&4&4&1&0\\2&9&8&2&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&4&4&1&0\\0&1&0&0&0\end{array}\right)\Rightarrow \begin{array}{l}x_1+4x_2+4x_3+x_4=0\\x_2=0\end{array}$$

$$\Rightarrow x_1=-4x_3-x_4$$

So we can completely describe S!

$$S=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-4x_3-x_4\\0\\x_3\\x_4\end{pmatrix}\right\}=\boxed{\left\{x_3\begin{pmatrix}-4\\0\\1\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \text{ s.t. } x_3,x_4\in\mathbb{R}\right\}}$$

(19) (A) If V, W are lines in $\mathbb{R}^3$ then $V^\perp, W^\perp$ are planes in $\mathbb{R}^3$. Two planes in $\mathbb{R}^3$ can't be orthogonal since they are either parallel or intersect in a line.

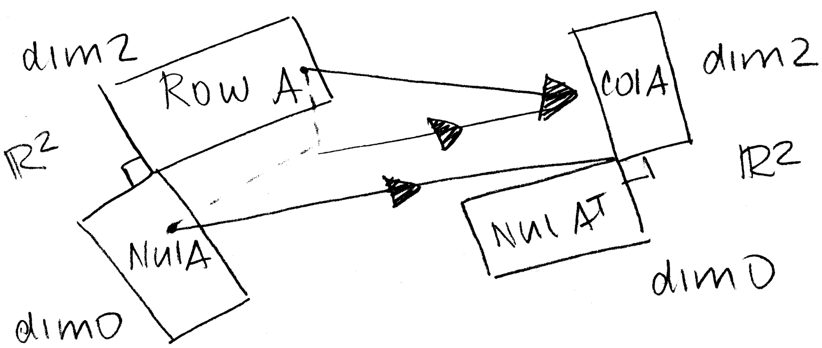
(B) Let $V = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$, $W = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, $Z = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

You check that $V \perp W$, $W \perp Z$, but V and Z are <u>not</u> orthogonal!

(22) Let $A = (1\ 1\ 1\ 1)$. Then $S = \text{Nul } A$.

$\Rightarrow S^\perp = (\text{Nul } A)^\perp = \text{Row } A$, $\boxed{\text{which has basis } \{(1\ 1\ 1\ 1)\}}$

(32) $A = \begin{pmatrix}1 & 2\\3 & 6\end{pmatrix}$



$B = \begin{pmatrix}1 & 0\\3 & 0\end{pmatrix}$