

5.1

$$\textcircled{4} \frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u, \quad u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\text{Let } u(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y \\ e^{\lambda t} z \end{pmatrix}$$

$$\text{Then } \frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u \Leftrightarrow \begin{pmatrix} \lambda e^{\lambda t} y \\ \lambda e^{\lambda t} z \end{pmatrix} = \begin{pmatrix} 1/2 e^{\lambda t} y + 1/2 e^{\lambda t} z \\ 1/2 e^{\lambda t} y + 1/2 e^{\lambda t} z \end{pmatrix}$$

$$\Leftrightarrow \lambda y = 1/2 y + 1/2 z$$

$$\lambda z = 1/2 y + 1/2 z$$

$$\Leftrightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u = \lambda u$$

So to find λ , look for eigenvalues of $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$:

$$\begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} = 0 \Leftrightarrow (1/2 - \lambda)^2 - 1/4 = 0$$

$$\Leftrightarrow 1/4 - \lambda + \lambda^2 - 1/4 = 0$$

$$\Leftrightarrow \lambda(\lambda - 1) = 0 \quad \Leftrightarrow \lambda = 0, \lambda = 1$$

Eigenvector for $\lambda_1 = 0$:

$$(A - 0I | 0) = \left(\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array} \right) \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 0$

Eigenvector for $\lambda_2 = 1$:

$$(A - 1I | 0) = \left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array} \right) \Rightarrow -x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = 1$.

\Rightarrow Complete solution: $u(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{1t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$t=0 \Rightarrow \text{need } c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 3 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -2 & -2 \end{array} \right)$$

#4, cont.

$$\Rightarrow c_1 + c_2 = 5$$

$$2c_2 = 8 \Rightarrow c_2 = 4, c_1 = 1$$

So our solution is

$$u(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or, writing the 2 components of u separately:

$$\begin{cases} v(t) = 1 + 4e^t & (v(0) = 5 \checkmark) \\ w(t) = -1 + 4e^t & (w(0) = 3 \checkmark) \end{cases}$$

⑤ $\lambda(A) = \{3, 1, 0\}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ corr. to $\lambda = 3$

$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ corr. to $\lambda = 1$

$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ corr. to $\lambda = 0$

$\text{Tr}(A) = 4$
 $\det(A) = 0$

$\lambda(B) = \{-2, 2, 2\}$

$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ corr. to $\lambda = -2$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ corr. to $\lambda = 2$

$\text{Tr}(B) = 2$
 $\det(A) = 8$

⑦ λ eval of A with e'vector x.

① WTS: x is also an e'vector of $B = A - 7I$

~~Pf~~ $Bx = Ax - 7Ix = Ax - 7x = \lambda x - 7x = (\lambda - 7)x$

\therefore x is an e'vector of B corr. to e'value $\lambda - 7$. \square

② Assume $\lambda \neq 0$.

WTS: x is also an e'vector of A^{-1} .

Pf $\lambda Ax = \lambda x \Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x)$

$\Rightarrow x = \lambda(A^{-1}x)$

$\Rightarrow \frac{1}{\lambda}x = A^{-1}x$

\therefore x is e'vector of A^{-1} corr. to e'value $1/\lambda$. \square

⑭ • $A = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{pmatrix}$

- $\Rightarrow \text{rank } A = 1$
- $\Rightarrow \dim(\text{Nul}(A)) = 3$
- $\Rightarrow 0$ is an eval of A with geo. multiplicity 3
- $\Rightarrow 0$ is an eval of A with algebraic multiplicity 3 or 4.

But $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \text{Tr}(A) = 4$
 $\Rightarrow \lambda_4 = 4.$

$\therefore \lambda(A) = \{0, 0, 0, 4\}$

Eigenspace corr. to $\lambda = 4$ is spanned by $\begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix}$.

• $C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

- $\Rightarrow \text{Rank}(C) = 2$
- $\Rightarrow \dim(\text{Nul } C) = 2$
- $\Rightarrow 0$ is an eval of A with geo. multiplicity 2
- $\Rightarrow 0$ is an eval of A with alg. multiplicity 2, 3, or 4
- can't be 3 since eigenvalues must sum to $\text{Tr}(C) = 0$.

$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} y+w \\ x+z \\ y+w \\ x+z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{pmatrix} \Rightarrow \lambda x = \lambda z \Rightarrow x = z$
 $\lambda y = \lambda w \ (\lambda \neq 0) \Rightarrow y = w.$

$\therefore \begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix}$ is an eigenvector of C .

Note $C \begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix} \therefore 2 \in \lambda(C)$

14, cont.

5.1

But if 2 is an e'val of C then so is -2 since eigenvalues must add up to 0 .

$$\boxed{\text{e'vals } \lambda(C) = \{2, -2, 0, 0\}}$$

$$\textcircled{25} \quad P = uu^T = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ \frac{5}{6} \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{3}{6} & \frac{5}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{36} & \frac{1}{36} & \frac{3}{36} & \frac{5}{36} \\ \frac{1}{36} & \frac{1}{36} & \frac{3}{36} & \frac{5}{36} \\ \frac{3}{36} & \frac{3}{36} & \frac{9}{36} & \frac{15}{36} \\ \frac{5}{36} & \frac{5}{36} & \frac{15}{36} & \frac{25}{36} \end{pmatrix}$$

$$\textcircled{a} \quad Pu = \underbrace{uu^T}_{\in \mathbb{R}} u = (u^T u) u = \underset{\substack{\downarrow \\ u \text{ is unit} \\ \text{vector}}}{1} u = u$$

$\therefore u$ is an e'vector of P corr. to $\lambda = 1$.

$$\textcircled{b} \quad v \perp u \Rightarrow v^T u = u^T v = 0$$

$$\therefore Pv = \underbrace{uu^T}_0 v = u(0) = 0$$

$\Rightarrow \lambda = 0$ is e'val of P with corr. e'vector v .

\textcircled{c} Find a basis for u^\perp (since by \textcircled{b} , all vectors in u^\perp are e'vectors corr. to $\lambda = 0$):

$$x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(27) \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad |\lambda I - P| = 0 \Leftrightarrow \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

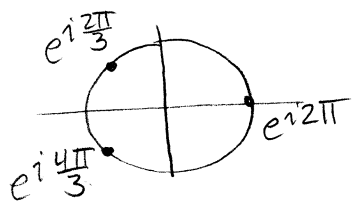
$$\Leftrightarrow \lambda \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda(\lambda^2) + (0 - (1)) = 0$$

$$\Leftrightarrow \lambda^3 - 1 = 0$$

$$\Leftrightarrow \lambda^3 = 1 = e^{i2\pi}$$

$$\Leftrightarrow \boxed{\lambda = e^{i2\pi/3}, e^{i4\pi/3}, e^{i2\pi}}$$



$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad |\lambda I - P| = 0 \Leftrightarrow \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & \lambda - 1 \\ -1 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda(\lambda - 1)(\lambda) - 1(0 - (\lambda - 1)(-1)) = 0$$

$$\Leftrightarrow \lambda^2(\lambda - 1) - 1(\lambda - 1) = 0$$

$$\Leftrightarrow (\lambda^2 - 1)(\lambda - 1) = 0$$

$$\Leftrightarrow (\lambda - 1)^2(\lambda + 1) = 0$$

$$\Leftrightarrow \boxed{\lambda = 1, 1, -1}$$