

5.1

$$\textcircled{4} \frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u, \quad u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\text{Let } u(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y \\ e^{\lambda t} z \end{pmatrix}$$

$$\text{Then } \frac{du}{dt} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u \Leftrightarrow \begin{pmatrix} \lambda e^{\lambda t} y \\ \lambda e^{\lambda t} z \end{pmatrix} = \begin{pmatrix} 1/2 e^{\lambda t} y + 1/2 e^{\lambda t} z \\ 1/2 e^{\lambda t} y + 1/2 e^{\lambda t} z \end{pmatrix}$$

$$\Leftrightarrow \lambda y = 1/2 y + 1/2 z$$

$$\lambda z = 1/2 y + 1/2 z$$

$$\Leftrightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} u = \lambda u$$

So to find λ , look for eigenvalues of $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$:

$$\begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} = 0 \Leftrightarrow (1/2 - \lambda)^2 - 1/4 = 0$$

$$\Leftrightarrow 1/4 - \lambda + \lambda^2 - 1/4 = 0$$

$$\Leftrightarrow \lambda(\lambda - 1) = 0 \quad \Leftrightarrow \lambda = 0, \lambda = 1$$

Eigenvector for $\lambda_1 = 0$:

$$(A - 0I | 0) = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 0$

Eigenvector for $\lambda_2 = 1$:

$$(A - 1I | 0) = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} \Rightarrow -x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = 1$.

\Rightarrow Complete solution: $u(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{1t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$t=0 \Rightarrow \text{need } c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 2 & 8 \end{pmatrix}$$