Math 102
Winter '08
Homework # 9

$\boxed{5.5}$

(16.) a) A real & sym $\Rightarrow$ A has all real eigenvalues

b) A stable $\Rightarrow$ $\text{Re}\,\lambda < 0 \quad \forall\, \lambda \in \Lambda(A)$

c) A orthogonal $\Rightarrow$ $|\lambda| = 1 \quad \forall\, \lambda \in \Lambda(A)$

d) A Markov $\Rightarrow |\lambda| \leq 1 \quad \forall\, \lambda \in \Lambda(A)$, and $\lambda = 1$ is an eigenvalue of A.

e) A defective $\Rightarrow$ A has at least one repeated eigenvalue

f) A singular $\Rightarrow$ $0 \in \Lambda(A)$.

(17) U, V unitary

Claim: UV is unitary

Pf WTS: $(UV)^H(UV) = I$

$$(UV)^H(UV) = V^H U^H U V = V^H I V \quad \text{since } U^H U = I$$
$$= V^H V$$
$$= I \quad \text{since } V \text{ unitary} \quad \boxed{\checkmark}$$

(18) (a) U unitary. Let U have eigenvalues $\lambda_1, \dots, \lambda_n$.
Then $|\lambda_1| = \cdots = |\lambda_n| = 1$ (Property 2')

$$\therefore |\det U| = |\lambda_1 \cdots \lambda_n|$$
$$= |\lambda_1| \cdots |\lambda_n|$$
$$= 1 \cdots 1$$
$$= 1 \qquad \checkmark$$

(b) Claim: In general, $\det U \neq \det U^H$

Pf: Just need a counterexample!

Take $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$

18 cont.

(c) Let $U = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be unitary.

Then $U^H U = I \Rightarrow \begin{pmatrix} \bar{w} & \bar{y} \\ \bar{x} & \bar{z} \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \bar{w}w + \bar{y}y & \bar{w}x + \bar{y}z \\ \bar{x}w + \bar{z}y & \bar{x}x + \bar{z}z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{aligned} &|w|^2 + |y|^2 = 1 \\ &|x|^2 + |z|^2 = 1 \\ &\begin{pmatrix} w \\ y \end{pmatrix}^H \begin{pmatrix} x \\ z \end{pmatrix} = 0 \end{aligned}}$$

(36) $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\sin^2\theta + \cos^2\theta = 1$
$\cos^2\theta - 1 = -\sin^2\theta$

$|Q - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\Leftrightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\Leftrightarrow \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0$

$\Leftrightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$

$\Leftrightarrow \lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \dfrac{\cancel{2}\cos\theta \pm \cancel{2}\sqrt{\cos^2\theta - 1}}{\cancel{2}}$

$\Leftrightarrow \lambda = \cos\theta \pm i\sin\theta$

$\Leftrightarrow \lambda = \cos\theta + i\sin\theta \boxed{= e^{i\theta}}$

OR $\lambda = \cos\theta - i\sin\theta = \cos\theta + i\sin(-\theta) \boxed{= e^{-i\theta}}$

E'vec. corr. to $\lambda = e^{i\theta}$ is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

E'vec. corr. to $\lambda = e^{-i\theta}$ is $\begin{pmatrix} 1 \\ i \end{pmatrix}$

(38) $v_1, \dots, v_n$ o.n. basis for $\mathbb{C}^n \Rightarrow$ matrix with those columns is an orthogonal matrix

Let $z \in \mathbb{C}^n$.

<u>claim</u> $z = (v_1^H z) v_1 + \cdots + (v_n^H z) v_n$.

Pf $z \in \mathbb{C}_n$, $\{v_1, \dots, v_n\}$ basis for $\mathbb{C}^n \Rightarrow$

$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_i \in \mathbb{C}^n$

$$\therefore v_1^H z = v_1^H \alpha_1 v_1 + \cdots + v_1^H \alpha_n v_n$$

$$= \alpha_1 (v_1^H v_1) + \cdots + \alpha_n \underbrace{(v_1^H v_n)}_{\to 0}$$

$$= \alpha_1 \quad \text{since} \quad v_1^H v_1 = 1 \text{ and } v_1^H v_j = 0 \ \forall j \neq 1$$

$\therefore v_1^H z = \alpha_1$

Similarly, $\alpha_i = v_i^H z \ \forall i$ ☑

(39) $\int_0^{2\pi} \overline{e^{-ix}} \cdot e^{ix} \, dx = 0$.

(41) A Hermitian $\Rightarrow (R + iS)^H = R + iS$

$\Rightarrow R^H - iS^H = R + iS$

$\Rightarrow R^H = R$, $S^H = -S$

$\Rightarrow R$ symmetric & S skew symmetric.

(42) The (complex) dimension of $\mathbb{C}^n$ is $\boxed{n}$

Nonreal basis for $\mathbb{C}^n$: $\underbrace{\left\{ \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i \end{pmatrix} \right\}}_{n \text{ vectors}}$