Math 102
Winter '08
Homework #10

$\boxed{6.1}$

4 a) $F(x,y) = -1 + 4(e^x - x) - 5x\sin y + 6y^2$ @ $(x,y) = (0,0)$

$$\frac{dF}{dx} = 4e^x - 4 - 5\sin y$$

$$\frac{d^2F}{dx^2} = 4e^x \Big|_{(0,0)} = 4$$

$$\frac{dF}{dy} = -5x\cos y + 12y$$

$$\frac{d^2F}{dy^2} = 5x\sin y + 12 \Big|_{(0,0)} = 12$$

$$\frac{d^2F}{dy\,dx} = -5\cos y \Big|_{(0,0)} = -5$$

$\therefore F_{xx}(0,0)\cdot F_{yy}(0,0) - (F_{yx}(0,0))^2 = 4(12) - 25 > 0$

Since $F_{xx}(0,0)$ also > 0, $F(0,0)$ is a $\boxed{\text{local min}}$

b) $F(x,y) = (x^2 - 2x)\cos y = x^2\cos y - 2x\cos y$ @ $(1,\pi)$

$$\frac{dF}{dx} = 2x\cos y - 2\cos y$$

$$\frac{d^2F}{dx^2} = 2\cos y \Big|_{(1,\pi)} = 2\cos\pi = -2$$

$$\frac{dF}{dy} = -x^2\sin y + 2x\sin y$$

$$\frac{d^2F}{dy^2} = -x^2\cos y + 2x\cos y \Big|_{(1,\pi)} = -\cos\pi + 2\cos\pi = 1 - 2 = -1$$

$$\frac{d^2F}{dy\,dx} = -2x\sin y + 2\sin y \Big|_{(1,\pi)} = -2\sin\pi + 2\sin\pi = 0$$

$-2(-1) - (0)^2 = 2 > 0$, $\frac{d^2F}{dx^2} < 0 \Rightarrow F(1,\pi)$ is

$\boxed{\text{local max}}$

$\boxed{6.2}$

② $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$|2| = 2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4+1 > 0$

$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= -(1-(-2)) + (-2-1) + 2(5)$

$= -3-3+10 > 0$

$\therefore$ A is P.D. by LB (III).

$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

$|2| = 2 > 0$

$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0$

$|B| = 4 > 0$

$\therefore$ B is P.D.

$C = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$

$|5| = 5 > 0$

$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} > 0$

$|C| = 16 > 0$

$\therefore$ C is P.D.

④ Eigenvalues of A^2 are squares of eigenvalues of A.
Eigenvalues of A^{-1} are reciprocals of e'vals of A.
$\therefore$ A has all pos. e'vals $\Rightarrow A^2, A^{-1}$ have all pos e'vals
i.e. A pos. def $\Rightarrow A^2, A^{-1}$ pos def.

6.2

(8) A spd, C n.s

<u>Claim</u>: $B = C^TAC$ is s.p.d.

(i) Show B symmetric:

$$B^T = (C^TAC)^T = C^TA^T(C^T)^T = C^T\underbrace{A^T}_{A}C$$

but A sym, so $A^T = A$

$\therefore$ $B^T = C^TAC = B$

i.e. B is symmetric ✓

(ii) Show B is pos def.

Let $x \in \mathbb{R}^n$. Then $x^TBx = x^TC^TACx$

$$= (Cx)^TA(Cx)$$

$$> 0 \quad \text{since } A \text{ pos def.}$$

Since x was arbitrary, we have that

$$x^TBx > 0 \quad \forall\, x \in \mathbb{R}^n$$

$\therefore$ B is pos def. ✓

(i) & (ii) $\Rightarrow$ B is s.p.d. ☑

(30) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

a) $|A| = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10.$

b) $\Lambda(A) = \{2, 5\}$

c) $A\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

$$\Rightarrow \left(A\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\; A\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right) = \left(2\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\; 5\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right)$$

eigenvectors of A.

30, cont.

d) A is pos. def b/c $\lambda > 0 \ \forall \lambda \in \Lambda(A)$.

A is symmetric since

$$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A. \checkmark$$

34) $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

pivot > 1 pivot > 1

$\therefore$ all pivots of A are > 1, but $\lambda = .5858$ is an e'value of A.

So the answer is $\boxed{\text{NO}}$.

$\boxed{6.3}$

2) a) $AA^T = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix}$

$\det(AA^T - \lambda I) = 0 \iff \begin{vmatrix} 17-\lambda & 34 \\ 34 & 68-\lambda \end{vmatrix} = 0 \iff (17-\lambda)(68-\lambda) - 1156 = 0$

$\iff \lambda = 0, 85$.

$\lambda = 0 = \sigma_2^2$: $\left(\begin{array}{cc|c} 17 & 34 & 0 \\ 34 & 68 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 17 & 34 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$\therefore \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is evec corr to $\lambda = 0$. $\left\|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\| = \sqrt{4+1} = \sqrt{5}$

$\therefore u_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

$\lambda = 85 = \sigma_1^2$: $\left(\begin{array}{cc|c} -68 & 34 & 0 \\ 34 & -17 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -68 & 34 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow -2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$

$\therefore \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is evec corr to $\lambda = 85$.

$\therefore u_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.