1. OLD EXAM PROBLEMS

NOTE These are exam problems from another course by another professor. There are some differences in style and emphasis. If confused, ask the TA or myself. While the final of this course will be somewhat different, practicing with these problems should be useful.

Problem 1.

Consider the following transport equation in the plane for the function $u(x, y)$:

$$\partial_x u + 2x\partial_y u \; = \; 0 \; . \tag{1}$$

a) Solve equation (1) with the following conditions on the y-axis:

$$u(0, y) \; = \; y^2 \; .$$

b) Is it possible to find a solution of (1) in the entire plane such that it has the following condition on the x-axis:

$$u(x, 0) \; = \; \sin(x) \; .$$

If not, give the reason why. (Hint: It may be helpful to sketch the characteristics of the equation (1)).

Problem 2.

This problem is close to something we discussed in a lecture. Consider the wave equation for a vibrating string of length $\ell = \pi$ with fixed endpoints (i.e. the Dirichlet problem). Suppose that the wave speed is $c = 1$ (i.e. we have that $T = \rho = 1$ for the tension and mass density). Suppose that the initial Cauchy data is given by:

$$\begin{aligned} u(0, x) \; &= \; \sin(x) + 2\sin(3x) - 5\sin(6x) + \sin(11x) \; , \\ u_t(0, x) \; &= \; -\sin(x) + 4\sin(7x) + \sin(10x) - 3\sin(15x) \; . \end{aligned}$$

Compute the energy at each time t for the wave $u(t, x)$. That is, compute the integral:

$$E(t) \; = \; \frac{1}{2}\int_0^\pi \Big(u_t^2(t, x) + u_x^2(t, x) \Big) \; dx \; .$$

(Hint: There is an easy way and a hard way to do this. Orthogonality and the conservation of energy are your friends.)

PROBLEM 4.

Suppose that one is *given* a solution $\Delta u = 0$ to the Laplace equation in the rectangle $0 < x < a$ and $0 < y < b$, with the Neumann boundary conditions:

$$\begin{aligned} -\partial_y u(x,0) &= f(x) \ , \\ \partial_x u(a,y) &= g(y) \ , \\ \partial_y u(x,b) &= k(x) \ , \\ -\partial_x u(0,y) &= h(y) \ . \end{aligned}$$

a) Prove that the boundary data f, g, k, h must satisfy the following integral identity:

$$0 = \int_0^a f(x)dx + \int_0^b g(y)dy + \int_0^a k(x)dx + \int_0^b h(y)dy \ .$$

Notice that this identity just represents the physically reasonable fact that for a steady state temperature distribution, the *total* flux across the boundary of the domain must be zero. If there was any net heat either coming in or going out of the rectangle, then the temperature distribution wouldn't be in equilibrium! (Hint: This is one of those identities that comes from directly manipulating the equation. Don't try to make some awful expansion).

b) Suppose that one were not aware of the conclusion of a) above, and that one naively tried to compute a solution to the problem:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } 0 < x < a \ , \ 0 < y < b \ , \\ \partial_y u(x,0) &= 0 \ , \\ \partial_x u(a,y) &= C \ , \\ \partial_y u(x,b) &= 0 \ , \\ \partial_x u(0,y) &= 0 \ , \end{aligned}$$

for some constant $C \neq 0$. *Assume* that the solution could only be a product $u(x,y) = X_0(x)Y_0(x)$ of two linear functions $X_0(x)$ and $Y_0(x)$. Why is it that one cannot find such a solution? That is, where does this procedure for finding a solution fail?

c) Write down an explicit solution $u(x,y)$ to the following "well-posed" Neumann problem:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } 0 < x < \pi \ , \ 0 < y < \pi \ , \\ \partial_y u(x,0) &= 0 \ , \\ \partial_x u(\pi,y) &= \cos(y) \ , \\ \partial_y u(x,\pi) &= 0 \ , \\ \partial_x u(0,y) &= 0 \ . \end{aligned}$$

PROBLEM 5.

Calculate an explicit solution to the following Dirichlet problem in a unit square with continuous boundary values:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } 0 < x < 1 \ , \ 0 < y < 1 \ , \\ u(x,0) &= 0 \ , \\ u(1,y) &= y \ , \\ u(x,1) &= 1 + \sin(\pi x) \ , \\ u(0,y) &= y \ . \end{aligned}$$

(Hint: Decompose this problem into two: One where you remove $\sin(\pi x)$ from the boundary conditions, and a second one where the boundary values are zero except for $u(x,1) = \sin(\pi x)$. It is possible to guess the solution in the first case, while we know how to do the second problem. Get the solution of the original problem from those two solutions).

PROBLEM 6.

a) Solve the following Dirichlet problem in the unit circle:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } 0 \leqslant r < 1 \ , \\ u(1,\theta) &= 1 + \sin(2\theta) + \cos(2\theta) \ . \end{aligned}$$

b) Verify that the maximum principle is true for this explicit solution. (Hint: It might be easiest to first subtract off a constant. Notice that if u is any function, then $u - C$ will have its maximum at the *same* point that u does, and that the maximum of $u - C$ is just the maximum of u minus C.)

PROBLEM 2.

Consider the solution $u(x,t)$ to the wave equation:

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u \ , \\ u(x,0) &= \begin{cases} \cos(x) \ , & \text{if } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases} \\ u_t(x,0) &= \begin{cases} \sin(x) \ , & \text{if } -\pi \leqslant x \leqslant \pi \ ; \\ 0 \ , & \text{otherwise} \ . \end{cases} \end{aligned}$$

Please compute the quantity $u(10,10)$.

PROBLEM 3.

Please compute the energy:

$$E[u](t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2+u_x^2)(x,t)\;dx\;,$$

of the solution $u(x,t)$ to the previous problem at time $t=10$.

PROBLEM 4.

The heat equation for a solid metal ring is the following ("periodic" boundary conditions):

$$\begin{aligned} u_t \;&=\; ku_{xx}\;, & -\pi\leqslant x\leqslant\pi\;,\\ u(-\pi,t)\;&=\;u(\pi,t)\;,\\ u_x(-\pi,t)\;&=\;u_x(\pi,t)\;,\\ u(x,0)\;&=\;f(x)\;. \end{aligned}$$

a) Use separation of variables to derive the general solution $u(x,t)$ to this problem.
b) Write down the specific solution $u(x,t)$ to the above problem with initial data:

$$f(x)\;=\;3+2\cos(3x)-5\sin(4x)\;.$$

c) Compute the steady state temperature $T_\infty=\lim_{t\to\infty}u(x,t)$ to the solution from part b). Show that this is equal to the average initial temerature:

$$T_{av}\;=\;\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\;dx\;.$$

PROBLEM 5.

Please compute the full sin/cos series on the interval $[-1,1]$ for the function:

$$\phi(x)\;=\;1-x^2\;.$$

(Hint: Notice that this function is even. Also, use integration by parts to compute the integrals.)

PROBLEM 6.

a) Solve the following Dirichlet problem in the unit circle:

$$\begin{aligned} \Delta u\;&=\;0\;, & \text{in }\;0\leqslant r<1\;,\\ u(1,\theta)\;&=\;1+3\sin(5\theta)+\cos(6\theta)-4\cos(10\theta)\;. \end{aligned}$$

b) Compute the average value:

$$A \ = \ \frac{1}{2\pi} \int_0^{2\pi} u(1,\theta) \ d\theta \ ,$$

and show that it is equal to $u|_{r=0}$.