WENZL'S RESEARCH

In the following a brief description of the main directions of Wenzl's research is given. It is then followed by short summaries of some of his papers. The numbers for references refer to the enclosed list of publications.

(i) *Period until promotion to full professor* Wenzl's research started out in the construction of examples of subfactors of the hyperfinite II_1 von Neumann factor. The idea is to find unitary factor representations of the braid group on infinitely many strands, and to obtain the subfactor from suitable subgroups. These constructions were carried out by analyzing unitary braid representations factoring through Hecke algebras (see [5]) and algebras found in collaboration with Joan Birman (see [3], and for the subfactor constructions, [12]), and independently by Murakami. The latter algebras are referred to here as q-Brauer algebras, and are also referred to as BMW-algebras. These algebras depend on one resp. two parameters. One of the difficulties in these constructions comes from the fact that the structure of these algebras is more complicated when one of these parameters is a root of unity - however, subfactors can only be constructed in these cases. These examples are closely connected to new (at that time) results in knot theory and turned out to be useful for constructing new invariants of 3-manifolds (see [14], [17]). Moreover, trace techniques from operator algebras also were useful in solving a long-standing problem about semisimplicity of Brauer's centralizer algebras (see [4]).

(ii) *Period after promotion to full professor* It was recognized that the results mentioned in part (i) can be appropriately described in the language of tensor categories - the latter can be understood as a generalization of the description of group symmetries and contains representation categories of groups as a special case. In particular, the cases treated in Wenzl's research described in part (i) correspond to certain deformations of representation categories of classical Lie groups, i.e. unitary, orthogonal and symplectic Lie groups. An important example of such categories are the representation categories of Drinfeld-Jimbo quantum groups, which are quantum deformations of universal enveloping algebras of Lie algebras depending on a parameter q. It was shown by H.H. Andersen that for q a root of unity the category of a special class of modules of quantum groups has a semisimple quotient with only finitely many irreducible objects up to equivalence; this quotient is referred to as a fusion category. The name comes from the fact that tensor categories equivalent to these quotient categories play an important role in several areas of mathematical physics. They have also found applications in mathematics such as the construction of topological invariants, as outlined below.

Subfactors: It was shown in [20] that for certain roots of unity these fusion categories have a C^*-structure. As an important consequence one obtaines a sequence of subfactors for each representation of a semisimple Lie algebra. In particular, the Jones subfactors and most of the examples in Wenzl's previous research appear in this construction as special cases. The approach essentially goes back to the one in [12]; however more advanced theory of quantum groups is necessary for the general case. This is so far the only publication which constructs these examples for all Lie types and all representations. In this context one should mention that an important analogous construction for type III_1 factors was carried out by Wassermann and his former student Toledano-Laredo via loop groups which so far has been published for the classical Lie types, and subfactors have also been constructed for certain representations of all Lie types except F_4 and E_8 by Feng Xu. It should be noted that all of these approaches are quite involved. This was one of the motivations for another approach in [27] to construct subfactors related to spinor representations via representations of braid groups of type B. This is similar to the comparatively elementary construction in [12], and also simplifies some of the steps there. Similarly, also the results of the E_N paper could be used to give an elementary construction of subfactors for type E_6 by explicitly constructing path representations of braid groups as it was done in the Hecke algebra paper.

Categories: In spite of the just mentioned results, the more elementary braid approach turned out to have several applications in its own right: The most important one deals with the classification of tensor categories via their Grothendieck semiring, i.e. via their tensor product rules. More precisely, in [15] and [30], a complete classification of all tensor categories is given, whose Grothendieck semirings are the one of the representation category of a classical Lie group G, i.e. for G a unitary, orthogonal or symplectic group. Moreover, this classification extends to fusion categories related to these groups. This is useful as there are a number of different ways to construct tensor categories using R-matrices, Hopf algebras or loop groups.

Another application of these techniques was found in Freedman's approach towards building a quantum computer which could implement Shor's algorithm of factoring numbers in polynomial time. He needed a complete classification of all possible tensor ideals of the Temperley-Lieb category, a tensor category which is closely related to the Lie group $SU(2)$. The solution of this problem was obtained by Wenzl in joint work with F. Goodman and was written up in an appendix to Freedman's paper (see [28]). Other applications include elementary constructions of modular tensor categories (which, in particular, produce invariants of 3-manifolds; see [19]) as well as the study of certain integrality structures of such categories (see [21]).

Braid representations and exceptional groups: The applications sketched in the last paragraph were only obtained for categories related to classical Lie groups. Another direction of Wenzl's research was to extend his braid type approach to categories related to spinor groups (see the already mentioned work [27]) and exceptional groups. For studying braid representations in connection with exceptional groups one only needs to consider representations for which the standard generators have at most 5 eigenvalues. As a first step, a complete classification of representations of the braid group B_3 with this property was given in [24], together with some applications for dimension formulas. Moreover, a detailed analysis of the combinatorics and the associated braid representations of an important part of tensor powers of certain representations of the quantum analog of the Kac-Moody algebras of type E_N was obtained in [29]. As an application, one obtains a so-called first fundamental theorem for tensor powers of the minuscule representations of quantum groups of type E_6 and E_7: it is shown that the commutant of the quantum group action is generated by the braid representation together with one additional generator. The analogous result for the representation categories of the corresponding Lie groups of type E_6 and E_7 follows from this and is also new even in this more classical setting.

Subgroup analogs and module categories It is well-known that although $SO(N)$ is a subgroup of $SU(N)$, this embedding can not be extended to an embedding of the corresponding Drinfeld-Jimbo quantum groups. Nevertheless, Wenzl has constructed subfactors which correspond, at least on a combinatorial level, to such subgroups on the level of fusion categories. The combinatorics of such subfactors is studied in [32]. The subfactors are constructed in the preprint *Fusion symmetric spaces and subfactors* via another q-deformation of Brauer's centralizer algebras (which is studied in detail in the preprint *A q-Brauer algebra*). These constructions seem to be closely related to certain co-ideal subalgebras of quantum groups, which were obtained by Letzter and Noumi, and to subfactors constructed by Feng Xu and Antony Wassermann via loop groups. The co-ideal algebra approach might make it possible to do this for all Lie types, similarly as it was done in [20] for fusion categories. This is currently being investigated.

Also, the joint paper [31] with Erlijman can be viewed as part of this program. Here an asymptotic construction is given for the construction of subfactors corresponding to the embedding of a braided C^* tensor category $\mathcal{C}$ into $\mathcal{C}^s$, for an integer $s > 0$. For the representation category of a finite group G, this would correspond to the diagonal embedding $g \in G \mapsto (g, g, \ \dots\ g) \in G^s$.

Here are short summaries of some of Wenzl's papers:

[3] *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. 313 (1989), joint work with Joan Birman. New quotient algebras of the group algebras of the braid groups are defined using Kauffman's link invariant. Their dimensions and decomposition in simple components in the generic case are determined. These algebras, also referred to as BMW-algebras, inspired and played an important role in Wenzl's further research (see [4], [12], [14], [17], [27], [30]) as well as with other authors. E.g. more recently, several authors considered BMW algebras of other Lie types (Häring-Oldenburg, A. Cohen and D. Wales, F. Goodman)

[4] *On the structure of Brauer's centralizer algebra* Ann. of Math. (2) 128 (1988). The Brauer algebras depend on a parameter x. Brauer showed that for $x = N$ they map surjectively on the commutant of the action of the orthogonal group $O(N)$ on the tensor powers of its N-dimensional representation. It is shown that these algebras are semisimple whenever x has nonintegral value. Moreover, an intrinsic description is given for the kernel of the map just mentioned for $x = N$ as the annihilator ideal of a certain trace functional.

[5] *Hecke algebras of type A_n and subfactors* Invent. Math. 92 (1988), Subfactors of the hyperfinite II_1 factor are constructed via unitary braid representations which factor through Hecke algebras of type A. The Hecke algebras depend on a parameter q. To obtain interesting subfactors, this parameter has to be a certain root of unity for which the structure of the Hecke algebras is quite complicated. Additionally, a useful estimate for the relative commutant of subfactors constructed via inductive limits of finite dimensional algebras is proved.

[10] *Littlewood-Richardson coefficients for Hecke algebras at roots of unity.* Adv. Math. 82 (1990), joint work with Fred Goodman. Using idempotents for Hecke algebra representations at roots of unity, quotients of the representation semiring of unitary groups are defined and their structure coefficients are computed in terms of the ones of the unitary groups.

[12] *Quantum groups and subfactors of type B, C, and D*, Comm. Math. Phys. 133 (1990): A classification of all unitary braid representations factoring over the q-Brauer algebra found in [3] is given. Indices and relative commutants of the resulting subfactors are computed. Here quantum groups are used for obtaining crucial data about unitarizibility of braid representations without having to deal with their very complicated representation theory for roots of unity.

[14] *Quantum invariants of* 3*-manifolds associated with classical simple Lie algebras*, Internat. J. Math. 4 (1993), joint work with Vladimir Turaev. It was shown by Reshetikin and Turaev that one can define invariants of 3-manifolds in connection with the Lie algebra sl_2 and a root of unity. The work in [3] and [5] is used in this paper to extend their approach to all classical Lie algebras.

[15] *Reconstructing monoidal categories*, Adv. Soviet Math., 16, Part 2 joint work with David Kazhdan: This paper shows that any monoidal semisimple rigid tensor category whose Grothendieck semiring is the one of a special unitary group $SU(N)$ must necessarily be equivalent to one of N possible twists of the representation category of the quantum group $U_q sl_N$ for q not a root of unity. This can be considered a generalization of a result by Drinfeld which says that there is essentially only one possible deformation of the representation category of $SU(N)$. Similar classification results are also proved for the above mentioned fusion categories in the $U_q sl_N$ case, for which one can not use deformation techniques.

[17] *Braids and invariants of* 3*-manifolds*, Invent. Math. 114 (1993) This paper gives another proof of the results in [14]. Here the invariants of 3-manifolds are obtained in a limiting process of certain link invariants.

[19] *Semisimple and modular categories from link invariants*, Math. Ann. 309 (1997), joint work with Vladimir Turaev: This gives an elementary construction of modular tensor categories

associated to orthogonal and symplectic groups. From these categories one obtains a wealth of topological invariants such as invariants of links and 3-manifolds, representations of mapping class groups and, more generally, topological quantum field theories.

[20] *C^*-tensor categories from quantum groups*, J. Amer. Math. Soc. 11 (1998): This paper shows that certain quotient categories of the representation categoy of Drinfeld-Jimbo quantum groups at roots of unity (in the followin also referred to as fusion categories), which were constructed by H.H. Andersen, can be supplied with a C^* structure for certain good roots of unity. As a consequence, one obtains for any representation of a semisimple Lie algebra a sequence of subfactors of the hyperfinite II_1 factor whose indices converge to the square of the dimension of the given representation. Additional invariants, called higher relative commutants, have also been computed. The Jones subfactors as well as various other examples of subfactors related to Hecke algebras and q-Brauer algebras appear as special cases of this general construction.

[21] *Integral modular categories and integrality of quantum invariants at roots of unity of prime order*, J. Reine Angew. Math. 505 (1998), joint work with Gregor Masbaum: It is shown that invariants of 3-manifolds related to classical Lie algebras take values in $\mathbf{Z}[\xi]$, where ξ is the root of unity of prime order for which the invariant is defined. This extended previous results for the sl_2 case. The motivation for this research comes from the so far not very well understood problem of finding and understanding good interpolations of the invariants when the parameter is not a root of unity.

[24] *Representations of the braid group B_3 and of $SL(2,\mathbf{Z})$*, Pacific J. Math. 197 (2001), joint work with Imre Tuba: A complete classification of all irreducible representations of the just mentioned groups is given up to dimension 5. Essentially, they are uniquely determined by the eigenvalues of one of the standard braid generators. As applications one obtains generic dimension formulas for representations of exceptional Lie algebras; moreover, this observation also proved useful in Wassermann's studies of fusion for Lie type E_8.

[27] *q-Centralizer algebras for spin groups* J. Algebra 253 (2002) joint work with Rosa Orellana: It is possible to study centralizer algebras involving a spinor representation in a similar uniform way as it was done before for the vector representations of classical groups. These algebras are quotients of the group algebras of braid groups of type B and depend on three parameters. In order to determine the unitarizability of these braid representations, it is necessary that a certain important trace functional has positive values on minimal idempotents. The main result of the paper is the calculation of these values in terms of these three parameters. The construction of subfactors can now be carried out along the lines of [12].

[29] *On tensor categories of Lie type E_N, $N \neq 9$*, Adv. Math. 177 (2003): It is well-known that the decomposition of tensor powers of the vector representations of the special unitary groups can be studied via the symmetric groups; a similar uniform behaviour is known for the decomposition of tensor powers of the vector representations of the orthogonal and symplectic groups, using Brauer's centralizer algebras. These uniform behaviours also carry over to the quantum group case. In this paper, it is shown that there exists a similar uniform behaviour for certain representations of Lie resp Kac-Moody algebras of Lie type E_N, $N \neq 9$. More precisely, it is shown that there exists a uniform decomposition behaviour for a certain nontrivial part of the tensor powers called the 'new part'. Moreover, it is shown that this new part carries a braid representation which essentially (i.e. up to one additional generator) generates the commutant of this new part, similarly as it was done before by Hecke algebras (for Lie type A) and by q-Brauer algebras (for Lie types BCD); in particular, it contains these algebras as quotients. This also has connections to the exceptional series of Lie algebras conjectured by Vogel and Deligne, which deals with the adjoint representations of exceptional Lie groups.

In addition, one obtains a first fundamental theorem for Lie types E_6 and E_7, as already mentioned in the general outline.

[30] *On tensor categories of type BCD*, joint paper with Imre Tuba, Crelle's Journal, 581 (2005) Here analogous results to the ones in [15] are proved for braided tensor categories whose Grothendieck semirings are isomorphic to the one of the representation category of an orthogonal or symplectic group. As before, they are essentially classified by a complex number q which can not be a root of unity except for 1, and two possible twists. Again, the techniques also apply to the associated fusion categories.

[31] *Subfactors from braided C^* tensor categories* In previous papers, Erlijman had constructed new examples of subfactors from unitary braid representations such as the ones in [5] and [12]. The subject of this paper is to further generalize the construction to any braided C^* tensor category. This contains two extreme cases which are comparatively well understood. In the classical case $\mathcal{C} = \mathrm{Rep}\ G$ for a finite group G, this corresponds to the embedding of $g \in G \mapsto (g, g,\ ..., g) \in G^s$, where $s > 1$. In the other extreme, we have a braided fusion category for which an important invariant derived from the braiding, called the S-matrix, is invariant (in the group case, the S-matrix only has rank 1). In the latter case, it is proved that the principal and dual principal graphs coincide, and can be calculated from the tensor product rules. In the group case, this is only true for the principal graph, while the dual graph is different. The dual principal graph is also calculated for other important examples with non-invertible S-matrix.

[32] *Quotients of representation rings* Many of the fusion categories can be considered to be quotients of representation categories of Lie groups; in particular, their Grothendieck semirings can be obtained via a quotient construction from the corresponding ones of Lie groups. In this paper, we observe that also the classical representation rings of full orthogonal and symplectic groups can be obtained as quotients of a formal representation category $Rep\ O(\infty)$. This allows us to calculate restriction coefficients from $Gl(N)$ to $O(N)$ from generic ones via an explicit quotient map. In particular, this approach can then be extended to similar restriction rules on the level of fusion categories which appear as multiplicities of principal graphs of certain subfactors. Those subfactors have been constructed in a recent preprint, *Fusion symmetric spaces and subfactors*, via another q-deformation of Brauer's centralizer algebra.

Research Articles (Hans Wenzl)

Unpublished work:

[1] Representations of Hecke algebras and subfactors, thesis, University of Pennsylvania (1985).

Published work:

[1] On sequences of projections, Math. Rep. C.R. Acad. Sc. Canada 9 (1987) 5-9.

[2] (with D. Handelman), Closedness of index values for subfactors, Proc. AMS 101 no.2 (1987) 277-282.

[3] (with J. Birman) Braids, link polynomials and a new algebra, Trans. AMS 313 (1989) 249-273.

[4] On the structure of Brauer's centralizer algebras, Annals of Math. 128 (1988) 173-193

[5] Hecke algebras of type A and subfactors, Invent. Math. 92 (1988) 349-383.

[6] (with P. de la Harpe), Operations sur les rayons spectraux de matrices symètriques entiéres positives, C.R. Acad. Sci. Paris, Serie I (1987) 733-736.

[7] Derived link invariants and subfactors, LMS Lecture Notes 136 (1988), D. Evans and M. Takesaki (editors), 237-240.

[8] Unitarizations of solutions of the quantum Yang-Baxter equation and subfactors, Proceedings of the Congress of the IAMP, Swansea 1988.

[9] Representations of braid groups and the quantum Yang-Baxter equation, Pacific Journal 145 (1990) 153-180

[10] (with F. Goodman) Littlewood-Richardson coefficients for Hecke algebras at roots of unity, Adv. Math, 82 (1990) 244-265

[11] (with A. Ram) Matrix units for centralizer algebras, J. of Algebra, 145 (1992) 378-395

[12] Quantum groups and subfactors of type B, C and D, Comm. Math. Phys. 133 (1990) 383-432

[13] Unitary braid representations, Int. J. of Modern Physics A, Vol. 7, Suppl. 1B (1992) 985-1006, Proceedings of the RIMS Research Project 1991 'Infinite Analysis' (survey article)

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[16] (with F. Goodman) The Temperley-Lieb algebra at roots of unity, Pacific J. of Math. 161 (1993) 307-334

[17] Braids and invariants of 3-manifolds, Invent. Math., 114 (1993) 235-275

[18] Subfactors and invariants of 3-manfolds, *Operator Algebras, Mathematical Physics and Low Dimensional Topology*, edited by Richard Herman and Betul Tanbay, Research Notes in Mathematics, A.K. Peters 1993

[19] (with V. Turaev) Semisimple and modular categories from link invariants, Math. Annalen 309 (1997), 411-461.

[20] C^* tensor categories from quantum groups, J. Amer. Math. Soc. 11 (1998), 261-282.

[21] (with G. Masbaum) Integral modular categories and integrality of quantum invariants at roots of unity of prime order, Crelle's Journal, 505, (1998), 209-235.

[22] (with F. Goodman) Iwahori-Hecke algebras of type A at roots of unity, J. of Algebra, 215 (1999) 694-734.

[23] (with F. Goodman) Crystal bases of quantum affine algebras and affine Kazhdan-Lusztig polynomials, Int. Math. Research Notes, (1999) No 5, 251-275.
[24] (with I. Tuba) Representations of the braid group B_3 and of $SL(2, \mathbf{Z})$, Pacific J. of Math, 197 (2001) 491-510.
[25] (with F. Goodman), A path algorithm for affine Kazhdan-Lusztig polynomials, Math. Z. 237, (2001) 235-249.
[26] Tensor categories and braid representations, LMS Lecture Note Series 290, 216-234.
[27] (with R. Orellana) q-Centralizer algebras for spin groups, J. of Algebra, 253 (2002), no. 2, 237–275.
[28] (with F. Goodman) Appendix to: Freedman, Michael H. A magnetic model with a possible Chern-Simons phase. Comm. Math. Phys. 234 (2003), no. 1, 129–183.
[29] On tensor categories of Lie type E_N, $N \neq 9$, Adv. Math. 177 (2003), no. 1, 66–104.
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[31] (with J. Erlijman) Subfactors from braided C^* tensor categories, Pacific J. Math, 231 (2007), 361–399.
[32] Quotients of representation rings, Representation Theory, 15 (2011) 385–406
[33] A q-Brauer algebra, preprint, arXiv:1102.3892
[34] On centralizer algebras for spin representations, preprint, arXiv:1107.4183
[35] Fusion symmetric spaces and subfactors, preprint, appears on archive August 9, 2011