FUSION SYMMETRIC SPACES AND SUBFACTORS

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ABSTRACT. We construct analogs of the embedding of orthogonal and symplectic groups into unitary groups in the context of fusion categories. At least some of the resulting module categories also appear in boundary conformal field theory. We determine when these categories are unitarizable, and explicitly calculate the index and principal graph of the resulting subfactors.

This paper is a sequel of our previous paper [W4], where we introduced a q-deformation of Brauer's centralizer algebra for orthogonal and symplectic groups; this algebra had already appeared more or less before in [Mo], see also discussion in [W4]. It is motivated by finding a deformation of orthogonal or symplectic subgroups of a unitary group which is compatible with the standard quantum deformation of the big group. This has been done before on the level of coideal subalgebras of Hopf algebras by Letzter. However, our categorical approach also allows us to extend this to the level of fusion tensor categories, where we find finite analogs of symmetric spaces related to the already mentioned groups. Moreover, we can establish C^* structures, necessary for the construction of subfactors, in this categorical setting; this is not so obvious to see in the setting of co-ideal algebras.

It is well-known how one can use a subgroup H of a (for simplicity here) finite group G to construct a module category of the representation category Rep G of G. This module category also appears in the context of subfactors of II_1 von Neumann factors as follows: Let R be the hyperfinite II_1 factor, and let $\mathcal{N} = R^G \subset \mathcal{M} = R^H$ be the fixed points under outer actions of G and H. Then the category of $\mathcal{N} - \mathcal{N}$ bimodules is equivalent to Rep G, and the module category is given via the $\mathcal{M} - \mathcal{N}$ bimodules of the inclusion $\mathcal{N} \subset \mathcal{M}$; its simple objects are labeled by the irreducible representations of H. In particular, an important invariant called the principal graph of the subfactor is determined by the restriction rules for representations from G to H. Important examples of subfactors were constructed from fusion categories whose Grothendieck semirings are quotients of the ones of semisimple Lie groups. So a natural question to ask is whether one can perform a similar construction in this context. More precisely, can we find restriction rules for type A fusion categories which describe a subfactor as before, and which will approach in the classical limit the usual restriction rules from $U(N)$ to $O(N)$.

We answer this question in the positive in this paper via a fairly elementary construction. We show that certain semisimple quotients of the q-Brauer algebras have a C^* structure and contain C^*-quotients of Hecke algebras of type A. The subfactor is then obtained as the closure of inductive limits of such algebras. Due to its close connection to Lie groups, we can

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give very explicit general formulas for its index and its first principal graph. Observe that the Lie algebra $\mathfrak{sl}_N$ decomposes as an $\mathfrak{so}_N$ module into the direct sum $\mathfrak{so}_N \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a simple $\mathfrak{so}_N$-module. Then the index can be expressed explicitly in terms of the weights of $\mathfrak{p}$, see Theorem 3.4. As before in the group case, it can be interpreted as the quotient of the dimension of the given fusion category by the sum of the squares of q-dimensions of representations of orthogonal or symplectic subgroups whose labels are in the alcove of a certain affine reflection group; however in our case, there is no corresponding tensor category for the denominator, and the q-dimensions differ from the ones of the corresponding quantum groups. Also, the restriction rules for the corresponding bimodules of this subfactor, the first principal graph, can be derived from the classical restriction rules via an action of the already mentioned affine reflection group, similarly as it was done before for tensor product rules for fusion categories. However, in our case, the affine reflection comes from the highest short root of the corresponding Lie algebra in the non-simply laced case; it is also different from the one for fusion categories in the even dimensional orthogonal case.

In the case corresponding to $O(2)$, we obtain the Goodman-de la Harpe-Jones subfactors for Dynkin diagrams D_n. We similarly also obtain a series of subfactors for other even-dimensional orthogonal groups. Perhaps a little surprisingly, there does not seem to be an analogous construction which would correspond to the group $SO(N)$ with N even. Our examples for the odd-dimensional orthogonal group and for symplectic groups seem to be closely related to examples constructed by Feng Xu [X] and Antony Wassermann [Wa] by completely different methods. Several explicit cases are discussed in detail at the end of the paper, as well as their connections to other approaches, coming from boundary conformal field theory, subfactors and tensor categories.

The first chapter mostly contains basic material from subfactor theory which will be needed later. In the second chapter we review and expand material on the q-Brauer algebra as defined in [W4], see also [Mo]. In particular, we define C^*-structures for certain quotients and use that to construct subfactors. The third chapter is mainly concerned with the finer structure of these subfactors, such as explicit closed formulas for the index and calculation of the first principal graph. The same techniques would also extend to other examples, such as the ones in [X].

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1. II$_1$ FACTORS

1.1. **Periodic commuting squares.** We will construct subfactors using the set-up of periodic commuting squares as in [W1]. More precisely, we assume that we have increasing sequences of finite dimensional C^* algebras $A_1 \subset A_2 \subset \ ...$ and $B_1 \subset B_2 \subset \ ...$ such that $A_n \subset B_n$ for all $n \in \mathbb{N}$. Let Λ_n resp $\tilde{\Lambda}_n$ be labeling sets for the simple components of B_n and A_n respectively. Let G_n be the inclusion matrix for $A_n \subset B_n$. If we write a minimal idempotent $p_\mu \in A_{n,\mu}$ as a sum of minimal mutually commuting idempotents of B_n, then the entry $g_{\lambda\mu}$ of G_n denotes the number of those idempotents which are in $B_{n,\lambda}$. We say that our

sequences of algebras are periodic with period d, if there exists an $n_0 \in \mathbb{N}$ such that for any $n > n_0$ we have bijections j between Λ_n and Λ_{n+d} as well as between $\tilde{\Lambda}_n$ and $\tilde{\Lambda}_{n+d}$ which do not change the inclusion matrices for $A_n \subset B_n$ as well as for $A_n \subset A_{n+1}$ and $B_n \subset B_{n+1}$. This means, in particular, that $g_{j(\lambda)j(\mu)} = g_{\lambda\mu}$ for all $\lambda \in \Lambda_n$, $\mu \in \tilde{\Lambda}_n$, $n > n_0$.

The trace functional defines inner products on the algebras A_n and B_n by $(b_1, b_2) = tr(b_1^* b_2)$. Let $e_{A_{n+1}}$ and e_{B_n} be the orthogonal projections onto the subspaces A_{n+1} and B_n of B_{n+1}. Then the *commuting square condition* says that $e_{A_{n+1}} e_{B_n} = e_{A_n} = e_{B_n} e_{A_{n+1}}$ for all $n \in \mathbb{N}$. Finally, we also note that the trace tr is uniquely determined on A_n and B_n by its weight vectors $\mathbf{a}_n$ and $\mathbf{b}_n$ which are defined as follows: Let p_μ be a minimal idempotent in the simple component of A_n labeled by μ. Then we define $a_{n,\mu} = tr(p_\mu)$, and $\mathbf{a}_n = (a_{n,\mu})_\mu$, where μ runs through a labeling set of the simple components of A_n. The weight vector $\mathbf{b}_n$ for B_n is defined similarly. The following proposition follows from [W1], Theorem 1.5 (where the matrix $G = (g_{\lambda\mu})$ defined here would correspond to the matrix G^t in [W1]):

Proposition 1.1. *Under the given conditions, we get a subfactor $\mathcal{N} \subset \mathcal{M}$ whose index $[\mathcal{M} : \mathcal{N}]$ is equal to $\|\mathbf{a}_n\|^2 / \|\mathbf{b}_n\|^2$ for any sufficiently large n. Moreover, we have $\sum g_{\lambda\mu} a_{n,\lambda} = [\mathcal{M} : \mathcal{N}] b_{n,\mu}$.*

1.2. **Special periodic algebras.** In general, it can be quite hard to determine finer invariants of the subfactors, the so-called higher relative commutants (or centralizers) from the generating sequence of algebras. However, under certain circumstances, this can become quite easy. We describe such a set-up. It is a moderate abstraction of an approach which has already been used before by a number of authors. The reader familiar with tensor categories and module categories should think of the algebras $A_n = \text{End}_{\mathcal{C}}(X^{\otimes n})$ and the algebras $B_n = \text{End}_{\mathcal{D}}(Y \otimes X^{\otimes n})$ for X an object in a C^* tensor category $\mathcal{C}$, and Y an object in a module category $\mathcal{D}$ over $\mathcal{C}$. In the following, we will make the following assumptions beyond the ones in the previous subsection:

1. The algebras A_n will be monoidal C^*-algebras. This means we have canonical embeddings of C^* algebras $A_m \otimes A_n \to A_{n+m}$ with multiplicativity of the trace, i.e. $tr(a_1 \otimes a_2) = tr(a_1)tr(a_2)$.
2. We have canonical embeddings $B_m \otimes A_n \to B_{n+m}$, again with multiplicativity of the trace.
3. We have the commuting square condition for the sequences of algebras $A_n \subset B_n$ and $1 \otimes A_{n-1} \subset A_n$.
4. There exists $d \in \mathbb{N}$ and a projection $p \in A_d$ such that $(1_m \otimes p) A_{m+d} (1 \otimes p) \cong A_m$ and $(1_m \otimes p) B_{m+d} (1 \otimes p) \cong B_m$ for all $m \in \mathbb{N}$.

Examples for this set-up will be given at the end of this section and in Section 2. Moreover, any finite depth subfactor $\mathcal{N} \subset \mathcal{M}$ (see e.g. [GHJ], [EK] for definitions) produces algebras for such a set-up as follows: Let $\mathcal{M}^{\otimes n} = \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \otimes_{\mathcal{N}} ... \otimes \mathcal{M}$ (n factors). Obviously, $\mathcal{M}^{\otimes n}$ is an $\mathcal{N} - \mathcal{N}$ as well as an $\mathcal{M} - \mathcal{N}$ bimodule. One can check that for $A_n = \text{End}_{\mathcal{N}-\mathcal{N}} \mathcal{M}^{\otimes n} \subset B_n = \text{End}_{\mathcal{M}-\mathcal{N}} \mathcal{M}^{\otimes n+1}$ the axioms above are satisfied; here the embedding is defined by letting the elements of A_n act on the second to $(n+1) - st$ factor of $\mathcal{M}^{\otimes n+1}$. It is also possible

to define these algebras in connection of relative commutants in the Jones tower of relative commutants (see e.g. [Bi] for details). Recall that for factors $\mathcal{N} \subset \mathcal{M}$ the relative commutant (or centralizer) $\mathcal{N}' \cap \mathcal{M}$ is defined to be the set $\{b \in \mathcal{M}, ab = ba \text{ for all } a \in \mathcal{N}\}$.

Lemma 1.2. *The subfactor $\mathcal{N} \subset \mathcal{M}$ generated from the sequences of algebras $1_m \otimes A_n \subset B_{n+m}$ has relative commutant B_m. The same statement also holds with B_{n+m} and B_m in the last sentence replaced by A_{n+m} and A_m.*

Proof. This is essentially the proof used for Theorem 3.7 in [W1]. Observe that by induction on r and assumption 4 above, we also have $(1_m \otimes p^{\otimes r}) X_{m+rd}(1_m \otimes p^{\otimes r}) \cong X_m$ for $X = A, B$. It follows from Theorem 1.6 of [W1] that the dimension of the relative commutant $\mathcal{N}' \cap \mathcal{M}$ is at most equal to the dimension of B_m. The claim follows from the fact that $B_m \otimes 1_n$ commutes with $1_m \otimes A_n$ for all n.

1.3. **Bimodules and principal graphs.** We calculate the first principal graph for subfactors constructed in our set-up, using fairly elementary methods from [W1] as well as the bimodule approach. The latter was first used in the subfactor context by Ocneanu, see e.g. [EK]. For the connection between bimodules and principal graphs, see [Bi] and for more details compatible with our notation, see also [EW]. While most of this section has already appeared before implicitly or explicitly, the presentation in our set-up might be useful also in other contexts.

Pick k large enough so that $m = kd > n_0$. Hence the inclusion matrices for $A_{rd} \subset B_{rd}$ coincide for all $r \geq k$ using the bijection of simple components as described in Section 1.1. Let Λ_m and $\tilde{\Lambda}_m$ be labelling sets for the simple components of B_m and A_m respectively. Let $\mathcal{N}$ and $\mathcal{M}$ be the factors generated by the increasing sequences of algebras A_n and B_n respectively, see Prop. 1.1 or Lemma 1.2, with the m there equal 0. Both of these factors have a subfactor $\tilde{\mathcal{N}}$ generated by the subalgebras $1_m \otimes A_n \subset A_{n+m} \subset B_{n+m}$. We now define for each $\lambda \in \tilde{\Lambda}_m$ an $\mathcal{N} - \tilde{\mathcal{N}}$ bimdule N_λ as follows: It is the Hilbert space completion of $\mathcal{N}p_\lambda$ with respect to the inner product induced by tr, where p_λ is a minimal idempotent in $A_{m,\lambda}$, the simple component of A_m labeled by λ, with obvious left and right actions by $\mathcal{N}$ and $\tilde{\mathcal{N}}$. To ease notation, we shall often refer to it as an $\mathcal{N} - \mathcal{N}$ bimodule, using the isomorphism between $\tilde{\mathcal{N}}$ and $\mathcal{N}$ given by the trace preserving maps $a \in A_n \mapsto 1_m \otimes a \in A_{n+m}$.

Similarly, we define $\mathcal{M} - \tilde{\mathcal{N}}$ bimodules M_μ for any $\mu \in \Lambda_m$ which are Hilbert space completions of $\mathcal{M}p_\mu$, where p_μ is a minimal idempotent in the simple component $B_{m,\mu}$ of B_m. Finally, we define the inclusion numbers b_μ^λ for elements $\lambda \in \tilde{\Lambda}$ and $\mu \in \Lambda_m$ as usual (see Section 1.1).

Lemma 1.3. *The bimodules N_λ and M_μ are irreducible $\mathcal{N} - \tilde{\mathcal{N}}$ resp $\mathcal{M} - \tilde{\mathcal{N}}$ bimodules. We have the decomposition $M_\mu \cong \bigoplus_\lambda b_\mu^\lambda N_\lambda$ as $\mathcal{N} - \tilde{\mathcal{N}}$ modules.*

Proof. This is well-known (see e.g. [EW] for more details). It follows from Lemma 1.2 that the endomorphism ring of the $\mathcal{M} - \tilde{\mathcal{N}}$ bimodule $\mathcal{M}$ is given by B_m. Hence the $\mathcal{M} - \tilde{\mathcal{N}}$ bimodules M_μ are simple, as p_μ was chosen to be a minimal idempotent in B_m. One shows similarly that also the N_λ's are simple $\mathcal{N} - \tilde{\mathcal{N}}$ bimodules.

Observe that $\dim_{\mathcal{N}} N_\lambda = tr(p_\lambda)$ and $\dim_{\mathcal{M}} M_\mu = tr(p_\mu)$ (see e.g. [J]). Now if p_λ is a minimal idempotent in A_m, it follows from the definitions that $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}} N_\lambda := \mathcal{M}p_\lambda$ is isomorphic as an $\mathcal{M} - \tilde{\mathcal{N}}$ bimodule to the direct sum $\oplus b_\mu^\lambda M_\mu$. By Frobenius reciprocity, see e.g. [EK], [Bi] it follows that the module N_λ appears with multiplicity b_μ^λ in $\mathcal{M}_\mu$, viewed as an $\mathcal{N} - \tilde{\mathcal{N}}$ bimodule. Hence the $\mathcal{N} - \tilde{\mathcal{N}}$ bimodule M_μ has a submodule which is isomorphic to $\bigoplus_\lambda b_\mu^\lambda N_\lambda$. But as M_μ has $\mathcal{N}$-dimension $[\mathcal{M} : \mathcal{N}]tr(p_\mu)$, it coincides with this submodule, by Proposition 1.1.

Theorem 1.4. *Let $\mathcal{N} \subset \mathcal{M}$ be the subfactor generated by sequences of algebras $A_n \subset B_n$ satisfying the conditions in Section 1.2. Then its first principal graph is given by the inclusion graph for $A_{kd} \subset B_{kd}$ for sufficiently large k.*

Proof. It is well-known that the first principal graph is given by the induction-restriction graph of $\mathcal{M} - \mathcal{N}$ and $\mathcal{N} - \mathcal{N}$ bimodules appearing in the tensor products $\mathcal{M}^{\otimes n}$, $n \in \mathbb{N}$, where $\mathcal{M}^{\otimes n} = \mathcal{M} \otimes_{\mathcal{N}} \mathcal{M} \otimes \ ... \ \otimes_{\mathcal{N}} \mathcal{M}$ (n factors), see [EK], [Bi]. Obviously, this graph does not change if we replace all $X - \mathcal{N}$ bimodules H in this setting by $X - q\mathcal{N}q$ bimodules Hq, for $X = \mathcal{M}, \mathcal{N}$ and q a nonzero projection in $\mathcal{N}$. The claim can now be shown for $q = p^{\otimes k}$ where k is chosen large enough so that $kd > n_0$, using Lemma 1.3.

Recall that many examples come from module tensor categories, where $A_n = \mathrm{End}_{\mathbb{C}}(X^{\otimes n})$ and $B_n = \mathrm{End}(Y \otimes (X^{\otimes n})$ for an object X in a tensor category $\mathbb{C}$ and an object Y in the module category $\mathcal{D}$ over $\mathbb{C}$. In this setting, the weight vectors of our trace are given by $a_{n,\lambda} = \tilde{d}_\lambda / x^n$ and $b_{n,\mu} = d_\mu / y x^n$ for positive quantities $d_\mu, \tilde{d}_\lambda, x$ and y. Then we have

Corollary 1.5. *Assuming the conditions for the trace weights as just given, we have subfactors $\mathcal{N} \subset \mathcal{M}_\mu$ with index $[\mathcal{M}_\mu : \mathcal{N}] = d_\mu^2 [\mathcal{M} : \mathcal{N}]$, with $\mathcal{N} \subset \mathcal{M}$ as in Theorem 1.4.*

Remark 1.6. There is also a second important invariant for $\mathcal{N} \subset \mathcal{M}$, the dual principal graph. It can be analogously defined as an induction-restriction graph between irreducible $\mathcal{M} - \mathcal{M}$ and $\mathcal{M} - \mathcal{N}$ bimodules appearing in the tensor powers $\mathcal{M}^{\otimes n}$. Its calculation is more difficult than the first principal graph. This is quite similar to the corresponding problem for subfactors coming from conformal inclusions and related constructions, see e.g [X1], [BEK], [EW]. We plan to study this problem in a future publication via suitable adaptions of techniques in those papers.

1.4. **The GHJ-construction.** We give a well-known and well-studied example for our current set-up, which was first constructed in [GHJ]. Let G be a matrix with nonnegative integer entries and norm less than 2. It is well-known that such matrices are classified by Coxeter graphs of type ADE. We assume that the columns of G are indexed by the even vertices, and the rows by the odd vertices. We define C^*-algebras B_n by $B_0 = \mathbb{C}^{v_e}$, and $B_1 = \oplus M_{d_j}$, where v_e is the number of even vertices, and the summands of B_1 are labelled by the odd vertices j, whose dimension d_j is equal to the number of even vertices to which j is connected. The embedding $B_o \subset B_1$ is given by the inclusion matrix G. Then we define recursively B_{n+1} via Jones' basic construction [J] for $B_{n-1} \subset B_n$. Here the trace on B_n is the unique normalized trace whose values on minimal idempotents are given by the Perron-Frobenius vector of

$G^t G$ or GG^t, depending on whether n is even or odd, and the vector is normalized such that $tr(1) = 1$. Then the algebra B_{n+1} is generated by B_n, acting on itself via left multiplication, and the orthogonal projection e_n onto the subspace B_{n-1} of B_n, with respect to the inner product coming from the trace. The algebra A_n is defined to be the subalgebra generated by the identity 1 and the projections e_i, $1 \leq i < 1$. It is well-known that these algebras satisfy the commuting square condition, that they are periodic with periodicity 2, and that the Jones projections e_i satisfy the conditions of the projection p in Section 1.2. This has already been shown in [GHJ].

2. q-Brauer algebras

2.1. **Definitions.** Fix $N \in \mathbb{Z}$ and let $[N] = (q^N - q^{-N})/(q - q^{-1})$, where q is considered to be a complex number. We denote by $H_n(q^2)$ the Hecke algebra of type A_{n-1}. It is given by generators $g_1, g_2, \ldots g_{n-1}$ which satisfy the usual braid relations and the quadratic relation $g_i^2 = (q^2 - 1)g_i + q^2$. The q-Brauer algebra $Br_n(N)$ is the complex algebra defined via generators $g_1, g_2, \ldots g_{n-1}$ and e and relations

  (H) The elements $g_1, g_2, \ldots g_{n-1}$ satisfy the relations of the Hecke algebra $H_n(q^2)$.
  (E1) $e^2 = [N]e$,
  (E2) $eg_i = g_i e$ for $i > 2$, $eg_1 = q^2 e$, $eg_2 e = q^{N+1}e$ and $eg_2^{-1}e = q^{-1-N}e$.
  (E3) $g_2 g_3 g_1^{-1} g_2^{-1} e_{(2)} = e_{(2)} = e_{(2)} g_2 g_3 g_1^{-1} g_2^{-1}$, where $e_{(2)} = e(g_2 g_3 g_1^{-1} g_2^{-1})e$.

It is easy to see that this algebra coincides with the algebra defined in [W4] after substituting q there by q^2, and e there by $q^{1-N}e$ (with the q of this paper); this is also compatible with the different definition of $[N]$ in [W4]. We have chosen this parametrization as it will make it easier to define a $*$ structure on it. More precisely, if $|q| = 1$, there exists a complex conjugate antiautomorphism $b \mapsto b^*$ on $Br_n(N)$ defined by

(2.1) $$e^* = e, \quad g_i^* = g_i^{-1}, \ 1 \leq i < n.$$

It is easy to check at the relations that this operation is well-defined.

2.2. **Molev representation.** We give a representation of our algebra $Br_n(N)$ in $\mathrm{End}(V^{\otimes n})$, where $V = \mathbb{C}^N$. For this we use the matrices used by Molev in [Mo] for the definition of his q-deformation of Brauers' centralizer algebra. His defining relations are slightly different from ours; but Molev has informed the author that our algebra satisfies the relations of his algebra. It turns out that also his matrices satisfy the relations of our algebras, which we will outline here. Let R be the well-known solution of the quantum Yang-Baxter equation for type A. For simplicity we will use this notation for what is often denoted as $\check{R}$. If E_{ij} are the matrix units for $n \times n$ matrices, we define the following elements in $\mathrm{End}(V^{\otimes 2})$:

$$R = \sum_i q E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + \sum_{i < j}(q - q^{-1})E_{ii} \otimes E_{jj},$$

and

$$Q = \sum_{i,j} q^{N+1-2i} \ E_{ij} \otimes E_{ij}.$$

Moreover, if $A \in \text{End}(V^{\otimes 2})$, we define the operator $A_i \in \text{End}(V^{\otimes n})$ by

$$A_i \;=\; 1_{i-1} \otimes A \otimes 1_{n-1-i},$$

where 1_k is the identity on $V^{\otimes k}$. Then we have the following proposition, all of whose essential parts were already proved in [Mo]. However, the relations for our algebras are slightly different, so we give some of the adjustments of the work in [Mo] to our context below.

Proposition 2.1. *The map $g_i \mapsto qR_{n-i}$ and $e \mapsto Q_{n-1}$ defines a representation Φ of $Br_n(N)$. It specializes to the usual representation of Brauer's centralizer algebra in $\text{End}(V^{\otimes n})$ for $q = 1$.*

Proof. Most of the relations are already known or are easy to check. E.g. it is well-known that the matrices qR_i satisfy the relations of the Hecke algebra $H_n(q^2)$. Relation $(E1)$ is checked easily, and also the relations in $(E2)$ are fairly straightforward to check. It suffices to check $(E3)$ for $n = 4$. For this observe that by [Mo], (4.16), we have

$$Q_3 R_2 R_3 R_1 R_2 Q_3 = Q_1 Q_3 + q^{N+1}(q - q^{-1}) Q_3 (R_1 + q^{-1} 1),$$

in our notation. Using the relation $R_i = R_i^{-1} + (q - q^{-1})1$ for the second and third factor of the left hand side, one derives from this that

$$Q_3 R_2^{-1} R_3^{-1} R_1 R_2 Q_3 = Q_1 Q_3.$$

To check relation $(E3)$, observe that

$$R_1 R_2 (v_i \otimes v_i \otimes v_j \otimes v_j) \;=\; R_3 R_2 (v_i \otimes v_i \otimes v_j \otimes v_j),$$

where (v_i) is the standard basis for $\mathbb{C}^N = V$. One derives from this that $R_2^{-1} R_1^{-1} R_3 R_2 Q_1 Q_3 = Q_1 Q_3$. Moreover, the same calculations above also work with R_i replaced by R_i^{-1} and Q_j replaced by its transpose Q_j^T. Hence one can show as before that $R_2 R_3 R_1^{-1} R_2^{-1} Q_1^T Q_3^T = Q_1^T Q_3^T$. Transposing this, using $R_i^T = R_i$ shows the last part of the claim. $\blacksquare$

2.3. **Quotients.** We can now rephrase the main results of [W4] in our notation as follows:

Theorem 2.2. *(a) There exists a well-defined functional tr on $Br_n(N)$ defined inductively by $tr(g_1) = q^{N+1}/[N]$, $tr(e) = 1/[N]$ and $tr(bg_n) = tr(b)tr(g_n)$ for all $b \in Br_n(N)$.*
(b) Let $\overline{Br}_n(N) = Br_n(N)/I_n$, where I_n is the annihilator ideal of tr. Then $\overline{Br}_n(N)$ is semisimple and the inclusion $\overline{Br}_n(N) \subset \overline{Br}_{n+1}(N)$ is well-defined for all n.

It is possible to explicitly describe the structure of the quotients $\overline{Br}_n = \overline{Br}_n(N)$. To do so, we need the following definitions for the labeling sets of simple representations. More conceptually, the labeling sets $\Lambda(N, \ell)$ consist of all such diagrams λ for which the quantities $d_\mu(q) \neq 0$ for any subdiagrams $\mu \subset \lambda$ including λ itself, where q^2 is a primitive ℓ-th root of unity and the d_μ's are defined in Section 2.5.

Definition 2.3. Fix integers N and ℓ satisfying $1 < |N| < \ell$.
(i) The set $\tilde{\Lambda}(N, \ell)$ consists of all Young diagrams with $\leq N$ rows such that the first and N-th row differ by at most $\ell - N$ boxes for $N > 0$. If $N < 0$, the Young diagrams have at most $|N|$ columns, where the first and $|N|$-th column differ by at most $\ell - |N|$ boxes.

(ii) The set $\Lambda(N, \ell)$ consists of all Young diagrams λ with λ_i boxes in the i-th row and λ'_j boxes in the j-th column which satisfy

(a) $\lambda'_1 + \lambda'_2 \leq N$ and $\lambda_1 \leq (\ell - N)/2$ if $N > 0$ and $\ell - N$ even,
(b) $\lambda'_1 + \lambda'_2 \leq N$ and $\lambda_1 + \lambda_2 \leq \ell - N$ if $N > 0$ and $\ell - N$ odd,
(c) $\lambda_1 \leq |N|/2$ and $\lambda'_1 + \lambda'_2 \leq \ell - |N|$ if $N < 0$ is even,
(d) $\lambda_1 + \lambda_2 \leq |N|$ and $\lambda'_1 + \lambda'_2 \leq \ell - |N|$ if $N < 0$ is odd.

Diagrams which miss one of these inequalities only by the quantity one are called boundary diagrams of $\Lambda(N, \ell)$; e.g. in case (a) if $\lambda'_1 + \lambda'_2 = N + 1$.

Theorem 2.4. *([W4], Section 5) Let q^2 be a primitive ℓ-th root of unity, and let N be an integer satisfying $1 < |N| < \ell$. Then the simple components of $\overline{Br}_n = \overline{Br}_n(N)$ are labeled by the Young diagrams in $\Lambda(N, \ell)$ with $n, n-2, n-4, \ldots$ boxes. If $V_{n,\lambda}$ is a simple $\overline{Br}_n$ module for such a diagram λ, it decomposes as a Br_{n-1} module as*

$$(2.2) \qquad V_{n,\lambda} \cong \bigoplus_{\mu} V_{n-1,\mu},$$

where μ runs through diagrams in $\Lambda(N, \ell)$ obtained by removing or, if $|\lambda| < n$, also by adding a box to λ.

2.4. Path idempotents and matrix units.

We will give some details about the proof of Theorem 2.4 which will also be needed for further results. Observe that the restriction rule 2.2 implies that a minimal idempotent p_μ in $\overline{Br}_{n-1,\mu}$ can be written as a sum of minimal idempotents with exactly one in $\overline{Br}_{n,\lambda}$ for each diagram λ in $\Lambda(N, \ell)$ which can be obtained by adding or subtracting a box from μ. This inductively determines a system of minimal idempotents and matrix units of $\overline{Br}_n(q^N, q)$ labeled by paths resp. pairs of paths in $\Lambda(N, \ell)$ of length n. Such a path is defined to be a sequence of Young diagrams $(\lambda^{(i)})_{i=0}^n$ where $\lambda^{(0)}$ is the empty Young diagram, and $\lambda^{(i+1)}$ is obtained from $\lambda^{(i)}$ by adding or removing a box. It follows from the restriction rule above that the dimension of $V_{n,\lambda}$ is equal to the number of paths of length n with $\lambda^{(n)} = \lambda$, and that we can label a complete system of matrix units for the simple component $\overline{Br}_{n,\lambda}$ by pairs of such paths. We then have the following lemma:

Lemma 2.5. *For each pair of paths t_1, t_2 in $\Lambda(N, \infty)$ with the same endpoint we can define the matrix unit E_{t_1,t_2} as a linear combination of products of generators over algebraic functions (rational for path idempotents) in q with poles only at roots of unity. More precisely, the formula for E_{t_1,t_2} is well-defined for q^2 a primitive ℓ-th root of unity if both t_1 and t_2 are paths in $\Lambda(N, \ell)$.*

Proof. This was proved in [W4], Section 5. As the result is not explicitly stated as such, we give some details here. One observes that the two-sided ideal generated by the element $\bar{e} \in \overline{Br}_{n+1}$ is isomorphic to Jones' basic construction for the algebras $\overline{Br}_{n-1} \subset \overline{Br}_n$ (or, strictly speaking, by certain conjugated subalgebras which are denoted by $i_1(\overline{Br}_n)$ and $i_2(\overline{Br}_{n-1})$, see Section 5.2 in [W4]). One can then define path idempotents and matrix units inductively as it was done in [RW], Theorem 1.4 using the formulas for the weights of traces, which will also

be reviewed in Section 2.5; this is closely related to what is also known in subfactor theory as the Ocneanu-Sunder path model [Su]. The complement of this ideal is a quotient of the Hecke algebra $\bar{H}_{n+1}$ for which matrix units already were more or less defined in [W1], p. 366.

Lemma 2.6. *Let $p_{[1^N]}$ be the minimal idempotent in H_N corresponding to its one-dimensional sign representation. Then we have $\bar{p}^{\otimes 2}_{[1^N]}\overline{Br}_{m+2N}\bar{p}^{\otimes 2}_{[1^N]} \cong \overline{Br}_m$ for all $m > 0$.*

Proof. Observe that if $p \in \overline{Br}_{2N,\emptyset}(N)$, the simple component labeled by the empty Young diagram $\emptyset$, then it follows from the restriction rule 2.2 (see also the equivalent version below Theorem 2.4) by induction on m that $p\overline{Br}_{m+2N}p \cong \overline{Br}_m$ for all $m \geq 0$. Hence it suffices to show that $p^{\otimes 2}_{[1^N]}$ is such an idempotent.

If $q = 1$ and $N > 0$, $\Phi(Br_n(N))$ coincides with the commutant of the action of the orthogonal group $O(N)$ on $V^{\otimes n}$, which is semisimple. Moreover, the trace tr is just a multiple of the pull-back of the natural trace on $\mathrm{End}(V^{\otimes n})$, so $\Phi(Br_n(N)) \cong \overline{Br}_n(N)$ at $q = 1$. As $\Phi(p_{[1^N]})$ projects onto the one-dimensional determinant representation in $V^{\otimes N}$, the claim follows easily in that case, using Brauer duality, i.e. the fact that $\Phi(Br_n(N))$ is equal to the commutant of $O(N)$ on $V^{\otimes n}$ for all n.

We will now use the fact that we can also define $Br_n(N)$ over the field of rational functions $\mathbb{C}(q)$, see [W4]. It follows from Lemma 2.5 that we can also define the path idempotents for $\overline{Br}_n(N)$ over that field for paths of length n in $\Lambda(N, \infty)$. As the rank of an idempotent is an integer, the claim follows as well for q a variable, and for $q \in \mathbb{C}$ not a root of unity. But as $p_t\bar{p}^{\otimes 2}_{[1^N]}p_t = 0$ for any path t of length $2N$ in $\Lambda(N, \ell)$ which ends in $\lambda \neq \emptyset$, we also get the rank 0 for $\bar{p}^{\otimes 2}_{[1^N]}$ at q^2 a primitive ℓ-th root of unity in $\overline{Br}_{2N,\lambda}(N)$. This finishes the proof for $N > 0$. The proof for the symplectic case $N < 0$ even goes the same way. □

2.5. **Weights of the trace.** Using the character formulas of orthogonal groups, one can calculate the weights of tr for the algebras $Br_n(N)$, i.e. its values at minimal idempotents of $Br_n(N)$. We will need the following quantities for a given Young diagram λ

$$(2.3) \qquad d(i,j) \;=\; \begin{cases} \lambda_i + \lambda_j - i - j & \text{if } i \leq j, \\ -\lambda'_i - \lambda'_j + i + j - 2 & \text{if } i > j. \end{cases}$$

Moreover, we define $h(i, j)$ to be the length of the hook in the Young diagram λ whose corner is the box in the i-th row and j-th column. We can now restate [W4], Theorem 4.6 in the notations of this paper as follows:

Theorem 2.7. *The weights of the Markov trace tr for the Hecke algebra $\bar{H}_n(q^2)$ are given by $\tilde{\omega}_\lambda = \tilde{d}_\lambda/[N]^n$, where $|\lambda| = n$, and for $\overline{Br}_n(N)$ they are given by $\omega_{\lambda,n} = d_\lambda/[N]^n$, where*

$$\tilde{d}_\lambda \;=\; \prod_{(i,j)\in\lambda} \frac{[N+j-i]}{[h(i,j)]}, \quad d_\lambda \;=\; \prod_{(i,j)\in\lambda} \frac{[N+d(i,j)]}{[h(i,j)]},$$

where λ runs through all the Young diagrams in $\tilde{\Lambda}(N, \ell)$ with n boxes for $\bar{H}_n(q^2)$, and through all Young diagrams in $\Lambda(N, \ell)$ with $n, n-2, n-4, \ldots$ boxes. for $\overline{Br}_n$.

Lemma 2.8. *The weights $\omega_{\lambda,n}$ are positive for all $\lambda \in \Lambda(N,\ell)$ if and only if $q^2 = e^{\pm 2\pi i/\ell}$ with $\ell > N$ and*
   (a) $N > 0$ and $\ell - N$ even or
   (b) $N < 0$ odd.

Proof. The weights can be rewritten for our choice of q as

$$\omega_{\lambda,n} \;=\; \frac{\sin^n(\pi/\ell)}{\sin^n(N\pi/\ell)^n} \prod_{(i,j)\in\lambda} \frac{\sin(N + d(i,j))\pi/\ell}{\sin(h(i,j)\pi/\ell)}.$$

As $h(i,j) \leq h(1,1) = \lambda_1 + \lambda_1' - 1 < \ell$ for all boxes (i,j) of λ, it follows that all factors in the formula above are positive for $N > 0$ (negative for $N < 0$) except possibly the ones in the numerator under the product. If $N > 0$ and $\ell - N$ odd, one checks that for the diagram $[\ell - N + 1)/2]$ we have $\omega_{\lambda,|\lambda|} < 0$. By the same argument, one shows that $\omega_{\lambda,|\lambda|} < 0$ for $\lambda = [(|N| + 1)/2]$ and $N < 0$. In the other two cases, one checks that $0 < |d(i,j)| < \ell$ for all boxes (i,j) of a diagram $\lambda \in \Lambda(N,\ell)$.

2.6. C^*-quotients.

Proposition 2.9. *If the weights $\omega_{\lambda,n}$ are positive for all $\lambda \in \Lambda(N,\ell)$, the star operation defined by $e^* = e$ and by $g_i^* = g_i^{-1}$ makes the quotients $\overline{Br}_n$ into C^* algebras.*

Proof. The proof goes by induction on n, with the claims for $n = 1$ and $n = 2$ easy to check. By [W4], the two-sided ideal I_{n+1} generated by e in $\overline{Br}_{n+1}$ is isomorphic to Jones' basic construction for $\overline{Br}_{n-1} \subset \overline{Br}_n$, see also the remarks before Lemma 2.6. In particular, this ideal is spanned by elements of the form $b_1 e b_2$, with $b_1, b_2 \in i_1(\overline{Br}_n)$, where $i_1(a) = \Delta_{n+1} a \Delta_{n+1}^{-1}$, with $\Delta = (g_1 g_2 ... g_{n-1})(g_1 ... g_{n-2}) ... g_1$. By induction assumption and properties of Jones' basic construction, this ideal has a C^* structure given by $(b_1 e b_2)^* = b_2^* e b_1^*$. This coincides with the $*$ operation defined before algebraically. It was shown in [W4] that $\overline{Br}_{n+1} \cong I_{n+1} \oplus \bar{H}_{n+1}$, where $\bar{H}_{n+1}$ is a semisimple quotient of the Hecke algebra H_{n+1} whose simple components are labeled by the Young diagrams $\lambda \in \Lambda(N,\ell)$ with $n+1$ boxes. All these simple representations satisfy the (k,ℓ) condition in [W1]. It follows from that paper that the map $g_i^* = g_i^{-1}$ induces a C^* structure for any such representation. This finishes the proof.

Theorem 2.10. *For each choice of N and ℓ with $q^2 = e^{\pm 2\pi i/\ell}$, and for each nonnegative integer m we obtain a subfactor $\mathcal{N} \subset \mathcal{M}$ with $\mathcal{N}' \cap \mathcal{M} = \overline{Br}_m$ and with index*

$$[\mathcal{M} : \mathcal{N}] \;=\; [N]^m \, \frac{\sum_{\mu \in \tilde{\Lambda}(N,\ell)} \tilde{d}_\mu^2}{\sum_{\lambda \in \Lambda(N,\ell)} d_\lambda^2},$$

with notations as in Section 2.5. Moreover, its first principal graph is given by the inclusion graph for $\bar{H}_{2Nk} \subset \overline{Br}_{2nk+m}$ for any sufficiently large k.

Proof. Let us first check conditions 1-4 in Section 1.2 with $A_n = \bar{H}_n$ and $B_n = \overline{Br}_n(N)$ for $q = e^{\pi i/\ell}$ and $1 < |N| < \ell$. Condition 1 is well-known and was checked in e.g.[W1]. Similarly, Cond. 2 follows from the results in [W4], using the map $b \otimes g_i \in \overline{Br}_m \otimes \bar{H}_n \mapsto bg_{m+i}$.

Condition 3 means that the conditional expectation from $\overline{Br}_{n+1}$ to $\overline{Br}_n$ maps $\bar{H}_{n+1}$ onto $\bar{H}_n$. But as any element of $\bar{H}_{n+1}$ can be written as a linear combination of elements of the form $ag_n b$, with $a, b \in \bar{H}_n$, we have for any $c \in \overline{Br}_n$ that

$$tr(ag_n bc) = tr(g_n)tr(abc) = tr(E_{\bar{H}_n}(ag_n b)c).$$

Hence the commuting square condition is satisfied for any four algebras of the type above. Finally, Condition 4 follows for $d = 2N$ and the projection $p = p_{[1^N]}^{\otimes 2r}$ from Lemma 2.6.

The periodicity condition for $\bar{H}_n$ was shown in [W1] by proving that $\bar{p}_{[1^N]} \bar{H}_{m+N} \bar{p}_{[1^N]} \cong \bar{H}_m$. This induces an injective map $\tilde{\Lambda}(N, \ell)_m \to \tilde{\Lambda}(N, \ell)_{m+N}$ by adding a column of N boxes to the given Young diagram which has to become surjective for sufficiently large m by definition of $\tilde{\Lambda}(N, \ell)$. The $2N$ periodicity for the algebras $\overline{Br}_n(N)$ follows similarly using Lemma 2.6, or see [W4].

3. S-MATRIX

We will need certain well-known identities, which can be found in [Kc], except for one case, which is a variation of the other ones. Because of this, we review the material in more detail. This might also be useful to the non-expert reader, as the identities needed here can be derived by completely elementary methods.

3.1. **Lattices.** Let $M \subset L \subset \mathbb{R}^k$ be two lattices of full rank. This means that they are isomorphic to $\mathbb{Z}^k$ as abelian groups, and each of them spans $\mathbb{R}^k$ over $\mathbb{R}$. Moreover, we assume that we have an inner product on $\mathbb{R}^k$ such that $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$ for all $\mathbf{x}, \mathbf{y} \in M$. We define the dual lattice M^* to be the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$ for all $\mathbf{x} \in M$; the dual lattice L^* is defined similarly. Obviously $M \subset L$ implies $L^* \subset M^*$. Finally, we also assume that $A = L/M$ is a finite abelian group. Then each $\gamma \in M^*$ defines a character of A via the map $e^\gamma : \mathbf{x} \in L \mapsto e^{2\pi i(\gamma, \mathbf{x})}$. In particular, one can identify the group dual of A with M^*/L^*. Define the matrix $\tilde{S} = \frac{1}{|L:M|^{1/2}}(e^\gamma(\mathbf{x}))$, where γ and $\mathbf{x}$ are representatives for the cosets M^*/L^* and L/M. Then $\tilde{S}$ is the character matrix of A up to a multiple and one easily concludes that it is unitary. More precisely, we can view it as a unitary operator between Hilbert spaces V and V^* with orthonormal bases labeled by the elements of L/M and M^*/L^* respectively.

3.2. **Weights of traces.** We will primarily be interested in lattices related to root, coroot and weight lattices of orthogonal and symplectic groups. We define the lattices

(3.1) $$Q = \{\mathbf{x} \in \mathbb{Z}^k, \ 2| \sum x_i\} \quad \text{and} \quad P = \mathbb{Z}^k \cup (\varepsilon + \mathbb{Z}^k),$$

where ε is the element in $\mathbb{R}^k$ with all its coordinates equal to $1/2$. Observe that $P^* = Q$ with respect to the usual scalar product of $\mathbb{R}^k$. Moreover, one can identify coroot and weight lattices of $\mathfrak{so}_{2k}$ or $\mathfrak{so}_{2k+1}$ with Q and P respectively. In particular, we define for any $\gamma \in P$ the functional $e^\gamma : \mathbb{R}^k \to \mathbb{C}$ by $e^\gamma(\mathbf{x}) = e^{2\pi i(\gamma, \mathbf{x})}$. The Weyl group of type B_k acts as usual via permutations and sign changes on the coordinates. Let $a_W = \sum_w \varepsilon(w)w$, where $\varepsilon(w)$ is the sign of the element w. Then the characters χ_λ for $\mathfrak{so}_{2k+1}$ resp for $\mathfrak{sp}_{2k}$ are given by

$\chi_\lambda = a_w(e^{\lambda+\rho})/a_w(e^\rho)$, where $\rho = (k+1/2-i)$ for $\mathfrak{so}_{2k+1}$ and $\rho = (k+1-i)$ for $\mathfrak{sp}_{2k}$, and W is the Weyl group of type B_k.

We will also need the somewhat less familiar character formulas for the full orthogonal group $O(N)$: Recall that the irreducible representations of $O(N)$ are labeled by Young diagrams λ with at most N boxes in the first two columns. $O(N)$-modules labeled by Young diagrams $\lambda \neq \lambda^\dagger$ restrict to isomorphic $SO(N)$-modules if and only if $\lambda_1' = N - (\lambda^\dagger)_1'$ and $\lambda_i' = (\lambda^\dagger)_i'$ for $i > 1$. Hence if $g = exp(\mathbf{x})$ is an element in $SO(N)$, it suffices to consider the quantities $\chi_\lambda(g) = \chi_\lambda(\mathbf{x})$ for λ with at most k rows for $N = 2k$ or $N = 2k+1$. We can now express the weights of Theorem 2.7 in terms of these characters; in fact the formulas in Theorem 2.7 were derived from these characters, see [Ko] and [W4].

Lemma 3.1. *Let d_λ, $\tilde{d}_\lambda$ be as in Theorem 2.7 for $q = e^{\pi i/\ell}$. Moreover, we define for $|N| = 2k$ or $N = 2k+1$ the vector $\check{\rho} \in \mathbb{R}^k$ by $\check{\rho} = ((|N|+1)/2-i)_i$. By the discussion above, it suffices to evaluate $\chi^{O(N)}(\check{\rho}/\ell)$ for Young diagrams λ with $\lambda_1' \leq N/2$, which will be assumed in the following.*

(a) If $N = 2k+1 > 0$, then $d_\lambda = \chi_\lambda^{O(N)}(\check{\rho}/\ell) = \chi_\lambda^{SO(N)}(\check{\rho}/\ell)$.

(b) If $N = 2k > 0$ and $\lambda_1' \leq k$, then $d_\lambda = m(\lambda) \det(\cos(l_j\check{\rho}_i)/\det(\cos(k-j)\check{\rho}_i)$, where $l_j = (\lambda+\rho)_j = \lambda_j + k - j$ and where $m(\lambda) = 2$ or 1, depending on whether λ has exactly k rows or not.

(c) If $N = -2k$, then $d_\lambda = (-1)^{|\lambda|}\chi_{\lambda^t}(\check{\rho}/\ell)$ for the symplectic character labelled by the transposed diagram λ^T.

(d) We have $\tilde{d}_\lambda = \chi_\lambda^{SU(N)}(\rho/\ell)$ for $N > 0$ and $\tilde{d}_\lambda = (-1)^{|\lambda|}\chi_{\lambda^T}^{SU(N)}(\rho/\ell)$ for $N < 0$, where $\rho = ((|N|+1)/2-i) \in \mathbb{R}^{|N|}$.

Proof. Observe that $\check{\rho}$ is the element ρ of the Cartan subalgebra of $\mathfrak{sl}_N$, viewed as an element of the Cartan subalgebra of the Lie subalgebra $\mathfrak{so}_N$ or $\mathfrak{sp}_N$, depending on the case. The proof now goes as e.g the proof of Theorem 4.6 in [W4], which is essentially the one of [Ko]. The fact that these arguments also work for the special quotients $\overline{Br}_n$ follows from the proof of [W4], Theorem 5.5.

Remark 3.2. Let Δ_+ be the set of positive roots of a semisimple Lie algebra and $|\Delta_+|$ be its cardinality. As usual, we can express the Weyl denominator in $\chi_\lambda(\check{\rho}/\ell)$ in product form as

$$(3.2) \qquad \Delta(\check{\rho}/\ell) = \prod_{\alpha>0}(e^{(\alpha,\check{\rho})\pi i/\ell} - e^{-(\alpha,\check{\rho})\pi i/\ell}) = (-i)^{|\Delta_+|}\prod_{\alpha>0} 2\sin((\alpha,\check{\rho})\pi/\ell).$$

3.3. **Usual S-matrices.** As usual, we pick as dominant chamber C_+ the regions given by $x_1 > x_2 > \ldots x_k > 0$ for Lie types B_k and C_k. We also choose the fundamental domains D with respect to the translation actions of M, M^*, L, L^* such that it has 0 in its center; here the lattices M and L will be certain multiples of the lattices P, Q or $\mathbb{Z}^k$ to be specified later. Let $\bar{P}_+$ be the intersection of M^* with the fundamental alcove $D \cap C_+$.

Observe that we also obtain a representation of the Weyl group W on the vector spaces V and V^*. Then it is easy to check that $a_W(V^*)$ has an orthonormal basis $|W|^{-1/2}a_w(e^\gamma)$, with $\gamma \in \bar{P}_+$, and we can define a similar basis $a_W(\mathbf{x})$ for $a_W(V)$. Let S be the matrix which

describes the action of $\tilde{S}_{|a_W(V)}$ with respect to that basis. Then it is not hard to check (and we will do a slightly more complicated case below) that its coeffecients are given by

$$(3.3) \qquad s_{\gamma,\mathbf{x}} = \frac{1}{|L:M|^{1/2}} \sum_w \varepsilon(w) e^{2\pi i (w.\gamma,\mathbf{x})}.$$

If L is the weight lattice of a simple Lie algebra, the entry $s_{\gamma,\mathbf{x}}$ is the numerator of Weyl's character formula for the dominant weight $\lambda = \gamma - \rho$, up to the factor $|L:M|^{-1/2}$. As the columns of the unitary matrix S have norm one, it follows that

$$(3.4) \qquad \sum_\lambda \chi_\lambda^2(\mathbf{x}) = \frac{|L:M|}{\Delta^2(\mathbf{x})},$$

where Δ is the Weyl denominator, and the summation goes over the dominant weights λ such that $\lambda + \rho \in \bar{P}_+$. We are now in the position to prove some cases of the following proposition:

Proposition 3.3. *Let $\Lambda(N,\ell)_{ev}$ be the subset of $\Lambda(N,\ell)$ consisting of Young diagrams with an even number of boxes. Then we have*

$$\sum_{\lambda \in \Lambda(N,\ell)_{ev}} d_\lambda^2 = \frac{\ell^k}{b(N)} \prod_{\alpha > 0} \frac{1}{4\sin^2(\alpha,\check{\rho})\pi/\ell},$$

where $\check{\rho} = ((|N|+1)/2 - i)$ and $\alpha > 0$ runs through the positive roots of $\mathfrak{so}_N$ for $N > 0$ and of $\mathfrak{sp}_{|N|}$ for $N < 0$ even, and $b(N) = 2$ for $N = 2k > 0$, and $b(N) = 1$ otherwise.

Proof. Let us consider the case $N = 2k + 1 > 0$, with P and Q as in 3.1. Let $L = \ell^{-1}\mathbb{Z}^k$ and let $M_1 = Q$ and $M_2 = \mathbb{Z}^k$. Then we have $M_1^* = P$, and $M_2^* = \mathbb{Z}^k$. Now observe that M_1^* is the weight lattice of $\mathfrak{so}_N$, and the elements $\gamma \in \bar{P}_+$ are in 1-1 correspondence with the dominant weights λ of $\mathfrak{so}_N$ satisfying $\lambda_1 \leq (\ell - N)/2$, via the correspondence $\gamma = \lambda + \rho$. Moreover, $|L : M_1| = 2\ell^k$. Hence it follows from Eq 3.4 that $\sum \chi_\lambda^2(\check{\rho}) = 2\ell^k/\Delta^2(\check{\rho})$. Playing the same game for the lattice M_2, we now only get the sum over the characters χ_λ^2 for which $\lambda + \rho$ is in $\mathbb{Z}^k$, which is only half as large as before. Hence also the sum over the characters χ_λ^2 for which $\lambda \in \mathbb{Z}^k$ has to have the same value. This sum coincides with the right hand side of the statement for $N > 0$ odd, by the restriction rules for $O(N)$ to $SO(N)$ (see Lemma 3.1 and its preceding discussion).

The symplectic case $N = -2k < 0$ goes similarly. Here we define $M \subset L = \ell^{-1}P$, and with $L^* = \ell Q \subset M^* = \mathbb{Z}^k$. Then it follows that $\sum d_\lambda^2 = 2\ell^k/\Delta^2(\check{\rho}/\ell)$, where the summation goes over all diagrams λ such that $\lambda^T \in \Lambda(N,\ell)$. Playing the same game for $M = P$ and $M^* = Q$, we get $\sum d_\lambda^2 = \ell^k/\Delta(\check{\rho}/\ell)$, where now the summation goes over all even, or over all odd diagrams in $\Lambda(N,\ell)$, depending on whether the sum of coordinates of $\rho = (k+1-i)$ is odd or even. In each case, we obtain that $\sum_{ev} d_\lambda^2 = \ell^k/\Delta(\check{\rho}/\ell)$. We have proved the proposition except for the case $N = 2k > 0$, for which we need a little more preparation.

3.4. **Another S-matrix.** We now consider a slight generalization of the above. Observe that we can define a second sign function $\tilde{\varepsilon}$ for $W = W(B_k)$ which coincides with the usual sign function on its normal subgroup $W(D_k)$, while we have $\tilde{\varepsilon}(w) = -\varepsilon(w)$ for $w \notin W(D_k)$. It is easy to see that also in this case we have $\tilde{\varepsilon}(vw) = \tilde{\varepsilon}(v)\tilde{\varepsilon}(w)$ for all $v, w \in W$. We define $\tilde{a}_W = \sum \tilde{\varepsilon}(w)w$, and also denote the corresponding operators on the various (quotient) lattices and on the vector spaces V and V^* by the same symbol. One observes that now we get an orthonormal basis for $\tilde{a}_W(V^*)$ of the form $\mathbf{b}_\gamma = |Stab(\gamma)|^{-1/2}|W|^{-1/2}\tilde{a}_W(e^\gamma)$, labeled by the elements of $\bar{P}_+$ which now consist of the $\gamma \in D$ such that $\gamma_1 > \gamma_2 > \ ... \ \gamma_k \geq 0$. Observe that $|Stab(\gamma)|$ is equal to 1 or 2, depending on whether $\gamma_k > 0$ or $\gamma_k = 0$. One similarly defines a basis for $\tilde{a}_W(V)$. Let $\mathbf{x}$ be such that $Stab(\mathbf{x}) = 1$, i.e. $x_k > 0$, and let $b_\mathbf{x} = |W|^{-1/2}\tilde{a}_W(\mathbf{x})$. Then, writing $M^*/\ell L^*$ as a collection of W orbits, we obtain

$$\tilde{S}\mathbf{b}_\mathbf{x} \quad = |W|^{-1/2} \sum_{\lambda \in \bar{P}_+} \sum_{v,w \in W} \frac{1}{|Stab_W(\gamma)|} \tilde{\varepsilon}(w)\tilde{s}_{v.\gamma, w.\mathbf{x}} v.\gamma$$

$$= \sum_{\lambda \in \bar{P}_+} \sum_v (\sum_w \tilde{\varepsilon}(w)\tilde{s}_{w.\gamma, \mathbf{x}} \frac{1}{|Stab_W(\lambda)|}) \tilde{\varepsilon}(v)v.\gamma.$$

where we replaced $\tilde{\varepsilon}(w)$ by $\tilde{\varepsilon}(v)\tilde{\varepsilon}(w^{-1}v)$, $\tilde{s}_{v.\lambda, w.\mathbf{x}}$ by $\tilde{s}_{w^{-1}v.\gamma, \mathbf{x}}$ and finally also substituted $w^{-1}v$ by w. We see from this that the coefficient of $v.\gamma$ is equal to 0 if γ has a nontrivial stabilizer except in the case when $\gamma_k = 0$. Hence it follows that $\tilde{S}$ maps $a_W(V)$ into $a_W(V^*)$. Taking bases $(\tilde{a}_W(\gamma))_{\gamma \in P_+}$ and $(\tilde{a}_W(\mathbf{x}))$, we see that $\tilde{S}_{|a_W(V)}$ can be described by the matrix $S = (s_{\gamma, \mathbf{x}})$ whose coefficients are given for $\mathbf{x}$ with trivial stabilizer by

$$(3.5) \qquad\qquad s_{\gamma, \mathbf{x}} = |Stab(\gamma)|^{-1/2}|L : M|^{-1/2} \sum_w \varepsilon(w)e^{2\pi i(w.\gamma, \mathbf{x})}.$$

3.5. **Squares of characters.** Using the discussion before and the formulas of Lemma 3.1 it is not hard to see that for N even and $\lambda_1' \leq N/2$ we can write

$$\chi_\lambda^{O(N)} = m(\lambda)\tilde{a}_W(e^{\lambda+\rho})/\tilde{a}_W(e^\rho),$$

where $m(\lambda) = 2$ or 1 depending on whether λ has exactly k rows or not. In particular, applying this to the trivial representation, we obtain $2\Delta(\rho) = \tilde{a}_W(e^\rho)$.

Let P and Q be as in 3.1, and set $L = \ell^{-1}P$ and $M = \mathbb{Z}^k$. Then $L^* = \ell Q \subset M^* = \mathbb{Z}^k$, and it is easy to see that all of these lattices are $W = W(B_k)$-invariant. Moreover, let $\check{\rho}/\ell = (k + 1/2 - i)/\ell \in \ell^{-1}P = M^*$. Then it follows for $N = 2k$ and ℓ even that

$$\sum_{\lambda \in \Lambda(N, \ell)} \chi_\lambda^2(\check{\rho}\ell) \ = \ \frac{1}{\Delta^2(\check{\rho}\ell)} \sum_{\lambda_{k+1}=0, \lambda_1 \leq (\ell-N)/2} (\tilde{a}_W(e^{\lambda+\rho})(\check{\rho})\ell)^2 \ = \ \frac{|L : M|}{2\Delta^2(\check{\rho})} \sum_\lambda s_{\lambda, \check{\rho}/\ell}^2.$$

Now observe that the matrix S is unitary and that $[L : M] = 2\ell^k$. Moreover, by e.g. Prop. 1.1 and Theorem 2.10 the square sum over odd diagrams must be equal to the square sum

over even diagrams. Hence we obtain for $N > 0$ even, and ℓ even that

$$(3.6) \qquad \sum_{\lambda \in \Lambda(N,\ell)_{ev}} d_\lambda^2 \;=\; \frac{\ell^k}{2\Delta^2(\check\rho)},$$

where $\Lambda(N,\ell)_{ev}$ denotes the set of diagrams in $\Lambda(N,\ell)$ with an even number of boxes. This finishes the last case of the proof of Proposition 3.3

3.6. **Calculation of index.** As usual, identify the Cartan algebra of $\mathfrak{sl}_N$ with the diagonal $N \times N$ matrices with zero trace. The embedding of the Cartan algebras of an orthogonal or symplectic subalgebra is given via diagonal matrices for which the $N + 1 - i$-th entry is the negative of the i-th entry, for $1 \le i \le N/2$. Hence, if ϵ_i is the $\mathfrak{sl}_N$ weight given by the projection onto the i-th diagonal entry, we have $(\epsilon_{N+1-i})_{|\mathfrak{so}_N} = (-\epsilon_i)_{|\mathfrak{so}_N}$, with a similar identity also holding for symplectic subalgebras. Using our description of coroot and weight lattices of orthogonal and symplectic Lie algebras as sublattices of $\mathbb{R}^k$, and defining ϕ_i to be the projection onto the i-coordinate, we see that $(\epsilon_{N+1-i})_{|\mathfrak{so}_N} = -\phi_i = (-\epsilon_i)_{|\mathfrak{so}_N}$. This allows us to describe the decomposition of $\mathfrak{sl}_N$ as an $\mathfrak{so}_N$ resp. $\mathfrak{sp}_N$ module as follows: We have

$$(3.7) \qquad \mathfrak{sl}_N = \mathfrak{so}_N \oplus \mathfrak{p}, \quad \text{resp.} \quad \mathfrak{sl}_N = \mathfrak{sp}_N \oplus \mathfrak{p}$$

where $\mathfrak{p}$ is the nontrivial irreducible submodule in the symmetrization of the vector representation of $\mathfrak{so}_N$, resp. $\mathfrak{p}$ is the nontrivial irreducible submodule in the antisymmetrization of the vector representation of $\mathfrak{sp}_N$. The nonzero weights $\omega > 0$ of $\mathfrak{p}$ coming from positive roots of $\mathfrak{sl}_N$ and the multiplicity $n(\mathfrak{p})$ of the weight 0 in $\mathfrak{p}$ are given by

(a) $2\phi_i, \phi_i$ and $\phi_i \pm \phi_j$ for $1 \le i < j \le k$ with $n(\mathfrak{p}) = k$ for $\mathfrak{so}_N$ with $N = 2k + 1$ odd,
(b) $2\phi_i$ and $\phi_i \pm \phi_j$ for $1 \le i < j \le k$ with $n(\mathfrak{p}) = k - 1$ for $\mathfrak{so}_N$ with $N = 2k$ even,
(c) $\phi_i \pm \phi_j$ for $1 \le i < j \le k$ with $n(\mathfrak{p}) = k - 1$ for $\mathfrak{sp}_{|N|}$ with $N = -2k < 0$ even.

Theorem 3.4. *The index of the subfactor $\mathcal{N} \subset \mathcal{M}$ obtained from the inclusions of algebras $\bar{H}_n(q) \subset \overline{Br}_n(q^N, q)$ is given by*

$$[\mathcal{M} : \mathcal{N}] = b(N)\ell^{n(\mathfrak{p})} \prod_{\omega > 0} \frac{1}{4\sin^2(\omega, \check\rho)\pi/\ell},$$

where the product goes over the weights $\omega > 0$ of $\mathfrak{p}$ coming from positive roots of $\mathfrak{sl}_N$, as listed above, $n(\mathfrak{p})$ is the multiplicity of the zero weight in $\mathfrak{p}$, and $b(N)$ and $\check\rho$ are as in Prop. 3.3.

Corollary 3.5. *If $q = e^{\pi i/\ell} \to 1$, the index $[\mathcal{M} : \mathcal{N}]$ goes to ∞ with asymptotics $\ell^{\dim \mathfrak{p}}$.*

Proof. We use Theorem 2.10, where the denominator has been calculated in Proposition 3.3. The numerator follows from a standard argument for S-matrices for Lie type A, see [Kc], versions of which have also been used in this section. For an elementary calculation, see [E].

Remark 3.6. It is straightforward to adapt our index formula to subfactors related to other fixed points $H = G^\alpha$ of an order two automorphism α of a compact Lie group G, up to some integer (or perhaps rational) constant $b(H, G)$. Again, $\mathfrak{p}$ would be the -1 eigenspace of the induced action of α on the Lie algebra $\mathfrak{g}$, and the same S-matrix techniques applied in this

section would go through. E.g. our formulas for $N = 3$ and ℓ odd coincide with the ones at the end of [X] for even level of $SU(3)$, up to a factor 3. This is to be expected as in our case only those diagrams appear in the principal graph (see next section) which also label representations of the projective group $PSU(3)$.

3.7. **Restriction rules and principal graph.** It follows from Theorem 1.4 that the principal graph of $\mathcal{N} \subset \mathcal{M}$ is given by the inclusion matrix for $\bar{H}_{2k} \subset \overline{Br}_{2k}$ for k sufficiently large. This still leaves the question how to explicitly calculate these graphs. Observe that in the classical case $q = 1$ these would be given by the restriction rules from the unitary group $U(N)$ to $O(N)$, for $N > 0$. Formulas for these restriction coefficients have been well-known, see e.g. [Wy] (see Theorems 7.8F and 7.9C), Littlewood's formula (see e.g. [KT], Section 1.5, and the whole paper for additional results). Another approach closely related to the setting of fusion categories can also be found in [W5].

Let $b_\mu^\lambda(N)$ be the multiplicity of the simple $O(N)$-module V_μ in the $U(N)$ module F_λ, for $N > 0$, where λ, μ are Young diagrams. It is well-known that for fixed Young diagrams λ and μ, the number $b_\mu^\lambda(N)$ will become a constant b_μ^λ for N large enough. Fix now also $\ell > |N|$. We define similar coefficients in our setting as follows: Recall that the simple components of $\bar{H}_n$ are labeled by the diagrams in $\tilde{\Lambda}(N,\ell)_n$ and the ones of $\overline{Br}_n$ by the diagrams in $\Lambda(N,\ell)$. We then define for $\lambda \in \tilde{\Lambda}(N,\ell)$ and $\mu \in \Lambda(N,\ell)$ the number $b_\mu^\lambda(N,\ell)$ to be the multiplicity of a simple $\bar{H}_{n,\lambda}$ module in a simple $\overline{Br}_{n,\mu}$ module.

In the following lemma the symbol χ_μ will also be used for the $O(N)$ character corresponding to the simple representation labeled by the Young diagram μ. Moreover, we also denote by $\overline{Br}_\infty$ the inductive limit of the finite dimensional algebras $\overline{Br}_n$ under their standard inclusions, for fixed N and ℓ.

Lemma 3.7. *(a) Each $g \in O(N)$ for which $\chi_\mu(g) = 0$ for all boundary diagrams μ of $\Lambda(N,\ell)$ defines a trace on $\overline{Br}_\infty$ determined by $tr(p_\mu) = \chi_\mu(g)/\chi_{[1]}(g)^n$, where p_μ is a minimal projection of $\overline{Br}_{n,\mu}$.*

(b) For given $\lambda \in \tilde{\Lambda}(N,\ell)_n$ the coefficients $b_\mu^\lambda(N,\ell)$ are uniquely determined by the equations $\chi_\lambda^{U(N)}(g) = \sum_\mu b_\mu^\lambda(N,\ell)\chi_\mu(g)$ for all g as in (a), where the summation goes over all diagrams μ in $\Lambda(N,\ell)$ with $n, n - 2, \ldots$ boxes.

Proof. The formula in statement (a) determines a trace on $\overline{Br}_n$ for each n. To show that these formulas are compatible with the standard embeddings we observe that a minimal idempotent $p_\mu \in \overline{Br}_{n,\mu}$ is the sum of minimal idempotents $e_\lambda \in \overline{Br}_{n+1,\lambda}$ where λ runs through all diagrams in $\Lambda(N,\ell)$ obtained by adding or removing a box to/from λ, see Eq 2.2 and the remarks below that theorem. Evaluating the traces of these idempotents and multiplying everything by $\chi_\lambda(g)^{n+1}$, equality of the traces is equivalent to

$$\chi_\mu(g)\chi_{[1]}(g) = \sum_\lambda \chi_\lambda(g).$$

By the usual tensor product rule for orthogonal groups, the left hand side would be equal to the sum of characters corresponding to *all* diagrams λ which differ from μ by only one box.

It is easy to check that this differs from the sum above only by boundary diagrams, for which the characters at g is equal to 0. This shows (a).

For (b), we first show that $tr(p_\lambda) = \chi_\lambda^{U(N)}(g)/\chi_{[1]}^{U(N)}(g)^n$ for $p_\lambda \in \bar{H}_{n,\lambda}$ a minimal idempotent and tr a trace as in (a). As the weight vector for $\overline{Br}_{n+2N}$ is a multiple of the one of $\overline{Br}_n$, for n large enough, the same must also hold for the weight vectors of $\bar{H}_n$ and $\bar{H}_{n+2N}$, by periodicity of the inclusions. Hence these weight vectors must be eigenvectors of the inclusion matrix for $\bar{H}_n \subset \bar{H}_{n+2N}$. As this inclusion matrix is just a block of the $2N$-th power of the fusion matrix of the vector representation for the corresponding type A fusion category, its entries must be given by $U(N)$ characters of a suitable group element. To identify these elements, it suffices to observe that the antisymmetrizations of the vector representation, labeled by the Young diagrams $\lambda = [1^j]$, $1 \leq j \leq N$, remain irreducible as $O(N)$ modules. This means the corresponding Hecke algebra idempotent remains a minimal idempotent also in $\overline{Br}_j$. Hence $tr(p_\lambda) = \chi_\lambda^{U(N)}(g)$ for $\lambda = [1^j]$ and $1 \leq j \leq N$. But as the antisymmetrizations generate the representation ring of $U(N)$, and also of the corresponding fusion ring, the claim follows for general λ. For more details, see e.g. [GW]

Recall that the coefficient $b_\mu^\lambda(N, \ell)$ can be defined as the rank of p_λ in an irreducible $\overline{Br}_{n,\mu}$ representation. So obviously the formula in the statement holds for any g as in (a). Examples for such g come from $exp(\mathbf{x})$ with $\mathbf{x} \in M^* = \ell^{-1}Q$ for which the character is given by the expression $\chi_\lambda(\mathbf{x})$ as in Section 2.5. As the columns of the orthogonal S-matrix are linearly independent, this would identify $SO(N)$ representations. If N is odd, the two $O(N)$ representations which reduce to the same $SO(N)$ representation are labeled by Young diagrams with opposite parities. Hence only one of them can occur in the decomposition of a given $U(N)$ representation. A similar argument also works in the symplectic case.

For N even, we can have two diagrams λ and $\lambda^\dagger$ with the same $SO(N)$ character, where one of them, say λ has less than k rows. They can be distinguished by elements $g \in O(N)\backslash SO(N)$ for which $\chi_{\lambda^\dagger}(g) = -\chi_\lambda(g)$. It is well-known that such elements g must have eigenvalues ± 1, and $\chi_\lambda(g)$ is given by the character formula for $Sp(2k-2)$ in the remaining $2k-2$ eigenvalues (see [Wy]). It follows from the invertibility of the S-matrix for $Sp(2k-2)$ at level $\ell/2 - k$ (see [Kc]) that we can identify those diagrams λ by evaluating $\chi_\lambda^{Sp(2k-2)}(\mathbf{x}/\ell)$ for $\mathbf{x} \in \mathbb{Z}^{k-1}$ with $\ell/2 > x_1 > x_2 > ... > x_{k-1} > 0$, and that those elements satisfy the boundary condition $\chi_\lambda^{Sp(2k-2)}(\mathbf{x}) = 0$ for any boundary diagram λ.

The lemma above is illustrated in the following section for a number of explicit examples. We can also give a closed formula for the restriction coefficients, using a well-known quotient map for fusion rings (even though in our case, the quotient ring does not correspond to a tensor category as far as we know). In the context of fusion rings, this is known as the Kac-Walton formula; for type A see also e.g. [GW]. In our case, we need to use a slightly different affine reflection group $\mathcal{W}$. In the orthogonal case $N = 2k$ and $N = 2k + 1$ it is given by the semidirect product of $\ell\mathbb{Z}^k$ with the Weyl group of type B_k. In the symplectic case, it is given by the semidirect product of ℓQ with the Weyl group of type B_k. As usual, we define the dot action of $\mathcal{W}$ on $\mathbb{R}^k$ by $w.\mathbf{x} = w(\mathbf{x} + \rho) - \rho$, where ρ is half the sum of the positive roots

of the corresponding Lie algebra, with the roots embedded into $\mathbb{R}^k$ as described above, and ε is the usual sign function for reflection groups. This can be extended to an action on the labeling set of $O(N)$ representations by identifying a Young diagrams with $\leq k$ rows with the corresponding vector in $\mathbb{Z}^k$, and by using the restriction rule from $O(N)$ to $SO(N)$ in the other cases. See also e.g. Lemma 1.7 in [W5] for more details.

Theorem 3.8. *With notations as above, the restriction multiplicity $b_\mu^\lambda(N, \ell)$ for $N = 2k+1 > 0$ and $N = -2k$ is given by*

$$b_\mu^\lambda(N, \ell) = \sum_{w \in \mathcal{W}} \varepsilon(w) b_{w.\mu}^\lambda(N).$$

If $N = 2k > 0$, we have to replace ε by $\tilde{\varepsilon}$ (see Section 3.4) in the formula above.

Proof. Looking at the character formulas, we see that an action of an element w of the finite reflection group on λ just changes the character by the sign of w. Moreover, by definition of the elements $\mathbf{x}$ we have that $\chi_\lambda(\mathbf{x}) = \chi_{\lambda+\mu}(\mathbf{x})$ for any $\mu \in M$. It follows that $\chi_{w.\lambda}(\mathbf{x}) = \varepsilon(w)\chi_\lambda(\mathbf{x})$ for all $\mathbf{x} \in M^*$ and $w \in \mathcal{W}$. Hence summing over the $\mathcal{W}$-orbits, we obtain for any $\mathbf{x} \in M^*$, $\lambda \in \tilde{\Lambda}(N, \ell)$ and $\mu \in \Lambda(N, \ell)$ that

$$\chi_\lambda^{U(N)}(\mathbf{x}) = \sum_\gamma b_\gamma^\lambda(N)\chi_\gamma = \sum_\mu (\sum_w b_{w.\mu}^\lambda(N))\chi_\mu.$$

The claim now follows from this and Lemma 3.7.

4. EXAMPLES AND OTHER APPROACHES

4.1. **The case $N = 2$:** This corresponds to the Goodman-de la Harpe-Jones subfactors for type $D_{\ell/2+1}$, where $\ell > 2$ has to be even. It follows from our theorem that the even vertices of the principal graph are labeled by the Young diagrams λ with an even number n of boxes, at most two rows and with $\lambda_1 - \lambda_2 \leq \ell - 2$; there are $(\ell - 2)/2$ such diagrams. Their dimensions are given by $\tilde{d}_k = [2k + 1]$, $0 \leq k < (\ell - 2)/2$.

Moreover, one checks that $\Lambda(2, \ell)$ consists of Young diagrams $[j]$ with $0 \leq j \leq \ell/2 - 1$ and of $[1^2]$, one column with 2 boxes, with dimensions $d_{[j]} = 2 \cos j\pi/\ell$ for $j > 0$ and dimension equal to 1 for the remaining cases (i.e. for $\emptyset$ and for $[1^2]$). The restriction rule (i.e. principal graph) follows from writing the dimensions as

$$\tilde{d}_k = 2 \cos \tilde{k}\pi/\ell \; + \; 2\cos(\tilde{k} - 2)\pi/\ell + \; ... \; + 1,$$

where $\tilde{k} = \min \{k, \ell/2 - k\}$. Indeed, this determines the graph completely except for whether to pick the diagram $\emptyset$ or $[1^2]$ for the object with dimension 1. It follows from the restriction rule $O(2) \subset U(2)$ that we take $\emptyset$ for j even, and $[1^2]$ for j odd. To calculate the index one can check by elementary means that $\sum_{\lambda \; even} d_\lambda^2 = \ell/2$. Moreover, it is well-known that the sum $\sum_{\lambda \; even} \tilde{d}_\lambda^2$ over even partitions for $\mathfrak{sl}_2$ is equal to $\ell/4 \sin^2 \pi/\ell$. Hence we obtain as index $[\mathcal{M} : \mathcal{N}] = 1/2 \sin^2 \pi/\ell$.

4.2. **The case** $N = 3$. It is also fairly elementary to work out this case in detail. Recall that by Weyl's dimension formula we have

$$\tilde{d}_\lambda = \frac{[\lambda_1 - \lambda_2 + 1][\lambda_2 - \lambda_3 + 1][\lambda_1 - \lambda_3 + 2]}{[1]^2 [2]}.$$

Now observe that the product of two q-numbers is given by the tensor product rules for $\mathfrak{sl}_2$, i.e. we have for $n \geq m$ that $[n][m] = [n + m - 1] + [n + m - 3] + ... + [n - m + 1]$. As an example, we have

$$\tilde{d}_{[4]} \; = \; \frac{[6][5]}{[2]} \; = \; \frac{[10] + [8] + [6] + [4] + [2]}{[2]} \; = \; [9] + [5] + [1],$$

i.e. the fourth antisymmetrization of the vector representation of $U(3)$ decomposes as a direct sum of the one-, five- and nine-dimensional representation of $SO(3)$. One similarly can show the well-known result that the adjoint representation of $SU(3)$, labeled by the Young diagram $[2, 1]$ decomposes into the direct sum of the three- and the five-dimensional representation of $SO(3)$, i.e. $\mathfrak{p}$ is the five dimensional representation of $SO(3)$. Hence we get from Theorem 3.4 that the index is equal to

$$[\mathcal{M} : \mathcal{N}] \; = \; \frac{\ell}{4^2 \sin^2(2\pi/\ell) \, \sin^2(\pi/\ell)}.$$

We note that here as well as in the other examples, the dimensions (i.e. entries of the Perron-Frobenius vectors) are given by $|\tilde{d}_\lambda|$ for even vertices, and by $\sqrt{[\mathcal{M} : \mathcal{N}]}|d_\mu|$ for odd vertices, with $\tilde{d}_\lambda$ and d_μ as in Lemma 3.1. To consider explicit examples, the first nontrivial case for $N = 3$ occurs for $\ell = 7$. We leave it to the reader to check that in this case the first principal graph is given by the Dynkin graph D_8. A more interesting graph is obtained for $\ell = 9$, see Fig 4.1. Here we have the three invertible objects of the $SU(3)_6$ fusion category, including the trivial object (often denoted as $*$) on the left; they generate a group isomorphic to $\mathbb{Z}/3$. The vertices with the double edge are labeled by the object corresponding to the 5-dimensional representation of $SO(3)$ and the diagram $[4, 2]$ for $SU(3)_6$. This is the only fixed point under the $\mathbb{Z}/3$ action given by the invertible objects (or, in physics language, the currents). It would be interesting to see whether one can carry out an orbifold construction in this context related to the one in [EK1].

4.3. **The case** $N = 4$. This is the first nontrivial and, apparently, new case corresponding to even-dimensional orthogonal groups. As we shall see, somewhat surprisingly, the corresponding construction for $SO(4)$ does not seem to work. We do the case with $\ell = 8$ in explicit detail. It is not hard to check that we already get the periodic inclusion matrix for $n = 12$. As we consider an analog of the restriction to $O(4)$ for which the determinant can be ± 1, we should, strictly speaking, consider a fusion category for $SU(4) \times \{\pm 1\}$. We shall actually use the Young diagram notation for representations of $U(4)$. For $n = 12$ we have the invertible objects labeled by $[3^4]$, $[4^3]$, $[5^2 1^2]$ and $[6 2^3]$ (i.e. e.g. the last diagram has six boxes in the first and two boxes in the second, third and fourth row). They generate a subgroup isomorphic to $\mathbb{Z}/4$. It follows from the $O(4)$ restriction rules that $[3^4]$ and $[5^2 1^2]$
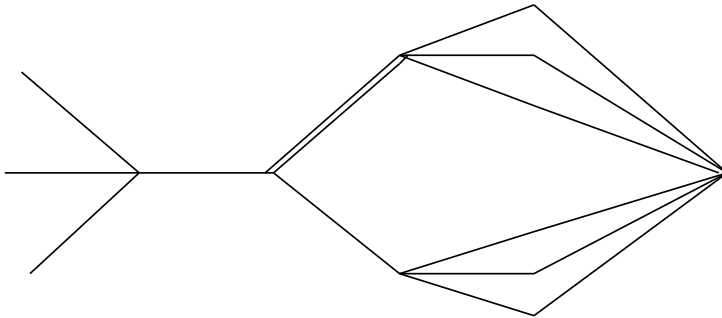
FIGURE 4.1. SO(3) for $\ell = 9$

contain the determinant representation, and $[4^3]$ and $[62^3]$ contain the trivial representation as one-dimensional $O(4)$ subrepresentations. This allows us to calculate the restrictions for representations of each $\mathbb{Z}/4$ orbit simultaneously. As usually for at least one element of each orbit the ordinary restriction rules still hold, it makes the general calculations easier. The principal graph can be seen in Fig. 4.2. As in the $N = 3$ example, the one-dimensional currents, including the trivial object $*$ appear as the left- and right-most vertices in the graph. The lowest vertex corresponds to the $O(4)$-object $[2]$ which is connected to the objects in the $\mathbb{Z}/4$-orbit $\{[2, 1^2], [3, 1], [4, 3, 1], [3, 3, 2]\}$. We also note that we get the same graph for the $Sp(4)$ case $N = -4$ for $\ell = 8$. However, for other roots of unities, already the indices of the subfactors differ which are given by

$$O(4) : \quad \frac{2\ell}{4\sin^2(3\pi/\ell) \; 4\sin^2(2\pi/\ell) \; 16\sin^4(\pi/\ell)} \qquad\qquad Sp(4) : \quad \frac{\ell}{4\sin^2(2\pi/\ell) \; 4\sin^2(\pi/\ell)}.$$

It was originally thought that we should also be able to get fusion category analogs for the restriction from $SU(N)$ to $SO(N)$ for N even. It is easy to check that this is not possible for $O(2)$. Some initial checks also seem to suggest a similar phenomenon for higher ranks. E.g. using the same element $\tilde{\rho}$ in the $SO(N)$ character formula would give dimension functions which are not invariant under the D_N diagram automorphism.

4.4. **Related results.** We discuss several results related to our findings. Our original motivation was to construct subfactors related to twisted loop groups. It was shown in R. Verill's PhD thesis [V] that it is not possible to construct a fusion tensor product for representations of twisted loop groups. However, it seemed reasonable to expect that representations of twisted loop groups could become a module category over representations of their untwisted counterparts. Many results, in particular about the combinatorics of such categories can be found in the context of boundary conformal field theory in papers by Gaberdiel, Gannon, Fuchs, Schweigert, diFrancesco, Petkova, Zuber and others (see e.g. [GG], [FS], [PZ] and the papers cited therein).
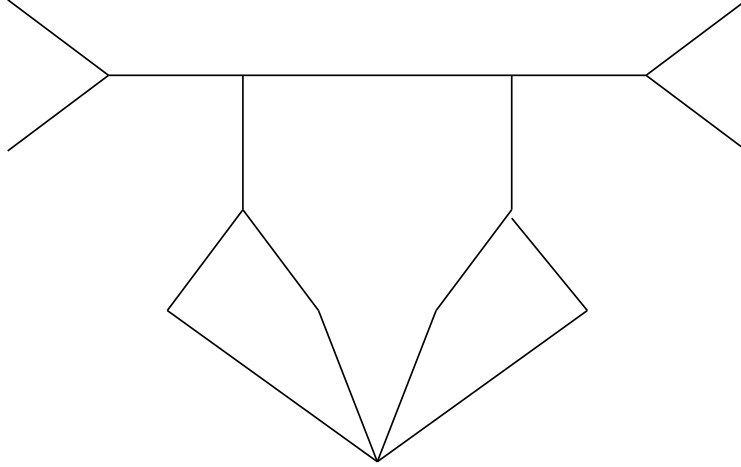
The principal graph for O(4) and Sp(4) with l=8

FIGURE 4.2. O(4) and Sp(4) for $\ell = 8$

In the mathematics literature, one can find closely related results in the papers [X] and [Wa]. Here the authors construct module categories via a completely different approach in the context of loop groups. E.g. the formulas at the end of [X] for the special case $N = 3$ differ only by a factor 3 (which can be explained, see Remark 3.6) from our formulas for $N = 3$ for even level (together with Corollary 1.5), modulo misprints; similar formulas for the symplectic case as well as restriction coefficients also appear at the end of [Wa]. We can not get results corresponding to the odd level cases in [X]. The combinatorics there suggests that this would require considering an embedding of $Sp(N-1)$ into $SU(N)$ under which the vector representation would not remain irreducible. In contrast, we can also construct module categories for $\ell-N$ odd, which would correspond to odd level; however, these categories are not unitarizable (which follows from Lemma 2.8) and they have different fusion rules. However, we do get fairly general formulas for the index and principal graphs of this type of subfactors in the unitary case, which was one of the problems posed in [X]. These formulas were known to this author as well as to Antony Wassermann at least back in 2008 when they had discussions about their respective works in Oberwolfach and at the Schrödinger Institute.

We close this section by mentioning that while our results for $N > 0$ odd and $N < 0$ even are in many ways parallel to results obtained via other approaches in connection with twisted loop groups, there does not seem to be an obvious analog for our results for $N > 0$ even. E.g. the combinatorial results in [GG] for that case seem to be different to ours.

4.5. **Conclusions and further explorations.** We have constructed module categories of fusion categories of type A via deformations of centralizer algebras of certain subgroups of unitary groups. We have also classified when they are unitarizable, and we have constructed the corresponding subfactors. These deformations are compatible with the Drinfeld-Jimbo deformation of the unitary group but not with the Drinfeld-Jimbo deformation of the subgroup. Most of the deformation was already done in [W4] via elementary methods. In principle, at least, it should be possible to use this elementary approach also for other inclusions. However, this might become increasingly tedious.

As we have seen already in Section 2.2, it should be possible to get a somewhat more conceptual approach using different deformations of the subgroup, see [N], [Mo], [L1], [L2], [IK] and references therein. In particular in the work of Letzter, such deformations via co-ideal algebras have been defined for a large class of embeddings of a semisimple Lie algebra into another one. At this point, it does not seem obvious how to define C^*-structures in this setting, and additional complications arise as these coideal algebras are not expected to be semisimple at roots of unity. Nevertheless, the results in this and other papers such as [X], [Wa] would seem to suggest that similar constructions might be possible also in a more general setting. It would also be interesting to see whether this could lead to a general solution of Problem 5.5 in [O].

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