# Restriction Coefficients for Classical Groups

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**Abstract**

A simple and fairly explicit formula has been given by Littlewood for the decomposition of a simple $GL(N)$-module if viewed as an $O(N)$ or $Sp(N/2)$-module for certain $GL(N)$-representations. We define an action of a reflection group on Young diagrams which allows us to compute the restriction multiplicities in the general case via an alternating sum over the Littlewood multiplicities.

There exist a number of ways how to describe the multiplicities of the restriction of a finite dimensional representation of $Gl(N)$ to $O(N)$ or $Sp(N/2)$. Classically, they are given by certain generating functions (see [Wy]). An explicit combinatorial formula in terms of well-known tensor product multiplicites of $Gl(N)$ was given by Littlewood. However, this formula only holds for $Gl(N)$ representations labeled by Young diagrams which are sufficiently 'small' in comparison to the rank $N$, and it is certainly not true in general. More recently, rather involved formulas have been found for the general case in [EW]. The purpose of this note is to give a fairly simple explicit formula for the general case. It is inspired by certain formulas for the tensor product multiplicities of so-called fusion categories, which is usually referred to as Kac-Walton formula. Similar as these fusion categories have Grothendieck semirings which are quotients of the ones of compact Lie groups, we show that the Grothendieck semiring of $O(N)$ or $Sp(N/2)$ is obtained as a quotient of a (formally defined) Grothendieck semiring $Gr(O(\infty))$, which has an $\mathbb{N}$-basis $\{[\lambda]\}$ labeled by the set of all Young diagrams. The quotient map can be explicitly described via an action of a reflection group $W$ of Coxeter type $D_\infty$ for orthogonal groups, and of type $B_\infty$ for symplectic groups on Young diagrams. Either the representation corresponding to a Young diagram $\lambda$ is already in the ideal, or it can be mapped via an element $w \in W$ to a diagram $w.\lambda$ in a fundamental domain which consists of a labeling set of irreducible representations of $O(N)$ resp. $Sp(N/2)$; we then have $[w.\lambda] \cong \varepsilon(w)[\lambda]$ modulo the above mentioned ideal, with $\varepsilon(w)$ the sign of $w$. The restriction multiplicities for a particular orthogonal or symplectic group can now be easily computed by taking Littlewood's ones modulo the above-mentioned ideal. An explicit formula is given in Theorem 3.2.

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# 1 Brauer algebras

## 1.1 Definitions

Brauer's centralizer algebra $\mathcal{C}_n$ of orthogonal type is defined over the field $\mathbb{C}(x)$ of rational functions over $\mathbb{C}$; the field $\mathbb{C}$ could be replaced as well by any field of characteristic 0, such as e.g. $\mathbb{Q}$. It has a basis given by graphs with $2n$ vertices, arranged in two lines, and $n$ edges such that each vertex belongs to exactly one edge. Multiplication of a graph $a$ with a graph $b$ is given by putting $a$ on top of $b$. The product is then defined to be the graph obtained by removing all cycles from the composite graph, multiplied by $x^c$, where $c$ is the number of cycles (see [Br]).

Moreover, we define for graphs $a \in \mathcal{C}_n$, $b \in \mathcal{C}_m$ a new graph $a \otimes b \in \mathcal{C}_{n+m}$ by putting the graph $b$ to the right of the graph $a$. Then $a^{\otimes k}$ is equal to $a \otimes a \otimes \ ... \ \otimes a$ ($k$ times) and $1_k$ is the identity for $\mathcal{C}_k$, the graph with $k$ vertical edges.

Observe that $\mathcal{C}_n$ contains the group algebra of the symmetric group $\mathbb{C}S_n$ as a subalgebra; it is spanned by graphs which connected the $i$-th lower vertex to an upper vertex, say the $\pi(i)$-th vertex, for $i = 1, 2, \ ... \ n$, where $\pi \in S_n$. Let $e$ be the graph in $\mathcal{C}_2$ with only horizontal edges. It is easy to see that $\mathcal{C}_n$ is generated by $S_n$ and $e_1 = e \otimes 1_{n-2}$.

Similarly, we define the algebra $\mathcal{C}_n(N)$ over $\mathbb{C}$ for any integer $N$ by substituting $x = N$. This specialization is important for describing the commutant $\mathrm{End}_G(V^{\otimes n})$ of the action of the orthogonal group $G = O(N)$ on $V^{\otimes n}$, where $V$ is the defining $N$-dimensional vector representation of $O(N)$. More precisely, one can define a surjective homomorphism $\Phi$ from $\mathcal{C}_n(N)$ onto $\mathrm{End}_G(V^{\otimes n})$. One can similarly define a symplectic Brauer algebra, and one obtains analogous homomorphisms for the symplectic group $Sp(N)$. It can be shown that the symplectic Brauer algebra is isomorphic to the orthogonal Brauer algebra; in particular, one obtains a surjective homomorphism from the orthogonal Brauer algebra with $x = -N$ onto $\mathrm{End}_G(V^{\otimes n})$ with $G = Sp(N/2)$.

It is easy to see that the tensor product operation defined for $\mathcal{C}_n$ above is compatible with these homomorphisms. Note also that the image of the element $e \in \mathcal{C}_2(N)$ in $\mathrm{End}_G(V^{\otimes 2})$ is a multiple of the projection onto the trivial representation in $V^{\otimes 2}$ corresponding to the bilinear form invariant under $G$.

The algebra $\mathcal{C}_n$ has a $q$-deformation in form of the algebra $\hat{\mathcal{C}}_n$, defined over the field $\mathbb{C}(r, q)$ of rational functions over $\mathbb{C}$ in variables $r$ and $q$ via generators and relations. The main reason we mention this algebra is that it can be easier described via generators and relations than the Brauer algebra itself. More precisely, it has generators $T_1, \ T_2 \ ... \ T_{n-1}$, which satisfy the braid relations $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and $T_iT_J = T_jT_i$ if $|i-j| > 1$ as well as the relations

$(R1)$ $\quad E_iT_i = r^{-1}E_i,$
$(R2)$ $\quad E_iT_{i-1}^{\pm 1}E_i = r^{\pm 1}E_i,$

where $E_i$ is defined by the equation

$$(D) \qquad (q - q^{-1})(1 - E_i) = T_i - T_i^{-1}.$$

It is possible to obtain the algebra $\mathcal{C}_n(N)$ as limit $\lim_{q\to 1} \hat{\mathcal{C}}_n(q^{N-1}, q)$.

## 1.2 Algebraic structure

As usual, a Young diagram $\lambda = (\lambda_i)$ is an array of boxes, with $\lambda_i$ boxes in the $i$-th row. We will freely identify $\lambda$ with a vector in $\mathbb{Z}^m$ whose $i$-th coordinate is equal to $\lambda_i$ whenever $m$ is larger than the number of boxes. Let $|\lambda|$ be the number of boxes of $\lambda$. We denote by $\lambda'$ the Young diagram with rows and columns interchanged. In particular, $\lambda_i'$ is the number of boxes in the $i$-th column. We can now describe the structure of the algebra $\mathcal{C}_n$ as follows:

(a) The algebra $\mathcal{C}_n$ is semisimple. Its simple components are labelled by the Young diagrams with $n$, $n-2$, $n-4$, ..., 1 resp. 0 boxes. The labeling is such that whenever $p_\lambda$ is a minimal idempotent in $C_{n,\lambda}$, then $p_\lambda \otimes ((1/x)e)^{\otimes r}$ is a minimal idemotent in $\mathcal{C}_{n+2r,\lambda}$.

(b) The decomposition of a simple $\mathcal{C}_n$ module $V_{n,\lambda}$ into simple $\mathcal{C}_{n-1}$ modules is given by

$$V_{n,\lambda} \cong \bigoplus_\mu V_{n-1,\mu}, \tag{1}$$

where the summation goes over all Young diagrams $\mu$ which can be obtained by either taking away or, if $\lambda$ has less than $n$ boxes, by adding a box to $\lambda$. The labeling of simple components is uniquely determined by the restriction rule, except for a possible choice of replacing $\lambda$ by its transposed $\lambda'$ simultaneously for all Young diagrams (see e.g. [W1], Lemma 2.11).

(c) The same statements hold for the algebra $\hat{\mathcal{C}}_n$. Here we have the convention that the eigenprojection of $T_1$ corresponding to its eigenvalue $q$ is labeled by the Young diagram [2].

(d) Statements (a) and (b) also hold for the $\mathbb{C}$-algebra $\mathcal{C}_n(N)$ provided that$|N| > n$. The cases with $n < |N|$ are more complicated. We need the following

**Definition 1.1** *(a) For $G = O(N)$ we define $D_N$ (resp. $I_N$) to be the set of all Young diagrams $\lambda$ satisfying $\lambda_1 + \lambda_2 \leq N$ (resp. those $\lambda$ for which $\lambda_1 + \lambda_2 = N + 1$ or $\lambda = [k]$, $k > N$).*

*(b) For $G = Sp(N/2)$ we define $D_N$ (resp. $I_N$) to be the set of all Young diagrams $\lambda$ satisfying $\lambda_1 \leq N/2$ (resp. $\lambda_1 = N/2 + 1$).*

*(c) The sets $D_N'$ and $I_N'$ are defined by analogous (in)equalities for the columns of Young diagrams. In particular, $\lambda \in D_N$ if and only if $\lambda' \in D_N'$*

(e) The restriction rules for the quotients $\bar{\mathcal{C}}_n(N) \cong \mathrm{End}_G(V^{\otimes n})$ are given as follows: The simple components of $\bar{\mathcal{C}}_n(N)$ are labeled by the Young diagrams in $D_N'$ with $n, n-2$, ...

boxes, and the restriction rule is as in (b), except that now only Young diagrams in $D'_N$ are allowed.

The simple restriction rule above allows us to define canonical minimal idempotents as follows: We call any sequence $t = (\lambda_{(i)})_{i=1}^n$ of Young diagrams a path of length $n$ if $\lambda_{(1)} = [1]$ and $\lambda_{(i+1)}$ is obtained from $\lambda_{(i)}$ by adding or subtracting a box. We denote by $t'$ the path obtained from $t$ by removing the last Young diagram. Then the path idempotent $p_t$ is defined to be the idempotent which acts on a simple $\mathcal{C}_n$-module $W_\lambda$ as 0 if $\lambda \neq \lambda_{(n)}$ and as $p_{t'}$ if $\lambda = \lambda_{(n)}$. Inductive formulas have been given for these idempotents in [RW]. We shall need the following result (see [RW], Theorem 2.3 and Theorem 2.4(b)):

**Proposition 1.2** *The path idempotent $p_t$ is given as a linear combination of the basis graphs whose coefficients are well-defined at $x = N$ whenever the path only contains Young diagrams in $D'_N$; this is also true if the last diagram of $t$ is in $I'_N$.*

## 1.3 $Gr(O(\infty))$

As an idempotent $p \in \mathcal{C}_n(N) \cong \mathrm{End}_G(V^{\otimes n})$ corresponds to the subrepresentation $pV^{\otimes n}$, we can translate the tensor product structure of $Rep(G)$ into the setting of the algebras $\mathcal{C}_n(N)$. More generally, this can also be done as well for the algebras $\mathcal{C}_n$, which will lead to the definition of a formal Grothendieck semiring $Gr(O(\infty))$.

We say that two idempotents $p$ and $q$ in an algebra $A$ are (conjugation) equivalent if there exist elements $u$ and $v$ in $A$ such that $p = uv$ and $q = vu$. More generally, we say that two idempotents $p \in \mathcal{C}_n$ and $q \in \mathcal{C}_m$ are equivalent if we can find nonnegative integers $n_1$ and $n_2$ such that $(\frac{1}{x}e)^{\otimes n_1} \otimes p$ and $(\frac{1}{x}e)^{\otimes n_2} \otimes q$ are equivalent as defined in the previous sentence in the algebra $\mathcal{C}_{n+2n_1} = \mathcal{C}_{m+2n_2}$. It follows from our labeling conventions that minimal idempotents in components of algebras $\mathcal{C}_n$ and $\mathcal{C}_m$ labeled by the same Young diagram are equivalent. We denote by $[p]$ the equivalence class of an idempotent $p$, and by $[\lambda]$ the equivalence class of a minimal idempotent in a simple component of $\mathcal{C}_{n,\lambda}$. In particular, if for $p \in \mathcal{C}_n$ we denote by $m_\lambda$ the trace of $p$ in the irreducible representation of $\mathcal{C}_n$ labeled by $\lambda$, we have $[p] = \sum_\lambda m_\lambda [\lambda]$. We define $Gr(O(\infty))$ to be the $\mathbb{N}$-span of $[\lambda]$, with $\lambda$ ranging over the set of all Young diagrams. It is not hard to show that this is equivalent to the set of all equivalence classes of idempotents in $\cup \mathcal{C}_n$.

If $p_\lambda$ and $p_\mu$ are minimal idempotents in $\mathcal{C}_n$ and $\mathcal{C}_m$ respectively, we obtain an idempotent $p_\lambda \otimes p_\mu$ in $\mathcal{C}_{n+m}$. If we define $d_{\lambda\mu}^\nu$ to be the rank of the idempotent $p_\lambda \otimes p_\mu$ in the irreducible representation of $\mathcal{C}_{n+m}$ labeled by the Young diagram $\nu$, we obtain

$$[\lambda][\mu] = [p_\lambda \otimes p_\mu] = \sum_\nu d_{\lambda\mu}^\nu [\nu].$$

By $\mathbb{N}$-linearity this extends to an associative and abelian multiplication on $Gr(O(\infty))$. We remark that we can define the same Grothendieck semiring by using the $q$-deformation $\hat{\mathcal{C}}_n$ of

$\mathcal{C}_n$. The latter is a consequence of the fact that the Drinfeld-Jimbo quantum group $U_q so_N$ has the same representation ring as the classical Lie algebra $so_N$.

Exactly the same construction also goes through if we replace $\cup\mathcal{C}_n$ by the union $\cup\mathbb{C}S_n$ of the group algebras of the symmetric group. In view of Schur duality, a minimal idempotent $p_\lambda \in \mathbb{C}S_n$ now corresponds to an irreducible representation $F^\lambda$ of $Gl(N)$ for $N$ sufficiently large. The resulting semiring will be denoted by $Gr(Gl(\infty)$. We denote the structure coefficients by $c_{\lambda\mu}^\nu$; which give the multiplicity of the simple $Gl(N)$-module $F^\nu$ in the tensor product $F^\lambda \otimes F^\mu$.

**Lemma 1.3** *Let $\lambda'$ denote the transpose of the Young diagram $\lambda$. Then the map $[\lambda] \mapsto [\lambda']$ defines an automorphism of $Gr(O(\infty))$ as well as of $Gr(Gl(\infty))$.*

*Proof*. It is easy to check that the generators of the algebra $\hat{\mathcal{C}}_n(r, q)$ also satisfy the relations of the algebra $\hat{\mathcal{C}}_n(r, -q^{-1})$. Hence the map $T_i \mapsto T_i$ induces an isomorphism between these two algebras. Due to our convention of labeling the simple components, the eigenprojection of $T_1 \in \hat{\mathcal{C}}_n(r, -q^{-1})$ for the eigenvalue $q$ is labeled by the Young diagram $[1^2]$. This implies that the simple component $\hat{\mathcal{C}}_{n,\lambda}(r, q)$ will be mapped to $\hat{\mathcal{C}}_{n,\lambda'}(r, -q^{-1})$, by uniqueness of the labeling of the simple components of $\hat{\mathcal{C}}_n$ up to transposition. This induces the desired automorphism of $Gr(O(\infty))$.

The claim is shown similarly for $Gr(Gl(\infty))$ by using the automorphism $s_i \mapsto -s_i$ of $\mathbb{C}S_n$, where $s_i$ is the transposition permuting $i$ with $i + 1$. $\Box$

Let us remark that the semiring $Gr(O(\infty))$ can also be interpreted as follows: If the number of boxes in the Young diagrams $\lambda$ and $\mu$ are small in comparison to $N$, the decomposition of the tensor product of the corresponding representations $V_\lambda$ and $V_\mu$ does not depend on the particular value of $N$. Hence $Gr(O(\infty))$ describes the tensor product rules of $O(N)$-representations labeled by Young diagrams $\lambda$ and $\mu$ whenever the number of boxes in these diagrams is small compared to $N$. Hence we can determine the multiplicative structure of $Gr(O(\infty))$ from the tensor product rules of orthogonal groups. For inductive proofs we define the *alphabetical order* on Young diagrams by $\lambda < \mu$ if and only if $\lambda_i < \mu_i$ for the smallest index $i$ for which $\lambda_i \neq \mu_i$. We can now formulate the following well-known results:

**Lemma 1.4** *(a) Let $\lambda$ be a Young diagram with $k$ boxes in its last column, and let $\tilde{\lambda}$ be the diagram $\lambda$ without its last column. Then $[\tilde{\lambda}][1^k] = \lambda + \sum m_\mu[\mu]$, where $\mu < \lambda$ in alphabetical order for all $\mu$ for which $m_\mu \neq 0$.*

*(b) Let $\lambda$ be a Young diagram with $k'$ boxes in its last column, and let $\tilde{\lambda}'$ be the diagram $\lambda$ without its last row. Then $[\tilde{\lambda}'][k'] = \lambda + \sum m_\mu[\mu]$, where $\mu' < \lambda'$ in alphabetical order for all $\mu$ for which $m_\mu \neq 0$.*

*Proof*. The proof follows from well-known tensor product rules for, say, $SO(2m + 1)$, as follows: First observe that it suffices to prove (a) in view of Lemma 1.3. Moreover, the set of weights of the highest weight module $V_{[1^k]}$ can be identified with the set of vectors in $\mathbb{Z}^m$ whose coordinates have absolute value 0 or 1, with at most $k$ nonzero coordinates. Moreover, it is a well-known consequence of the Weyl character formula that $V_\mu \subset V_\lambda \otimes V_\nu$ for simple modules with highest weights $\mu$, $\lambda$ and $\nu$ respectively only if $\mu = \lambda + \omega$, where $\omega$ is a weight of $V_\nu$. Identifying the diagram $\tilde{\lambda}$ with the corresponding vector in $\mathbb{Z}^m$ which represents the highest weight, the claim follows from the discussion above. □

The next proposition will be important when we compare the algebras $\mathcal{C}_n$ with the algebras $\mathcal{C}_n(N)$. While the latter algebras are not semisimple in general, we can still define a Grothendieck semiring as before, which we denote by $Gr(\mathcal{C}(N))$. The following proposition will be useful for comparing these two semirings. We formulate it in a slightly more general context as follows: Let $\mathcal{A}$ be a finite dimensional algebra defined over $\mathcal{C}(x)$ via a given basis $(b_i)$ for which the structure coefficients are well-defined at a given number $N \in \mathbb{C}$. Hence we can also consider the $\mathbb{C}$-algebra $\mathcal{A}(N)$ defined by the basis $(b_i(N))$ with the structure coefficients being the ones of $\mathcal{A}$ evaluated at $N$. It is easy to check that the algebras $\mathcal{C}_n$ satisfy these conditions for any $N \in \mathbb{C}$. Then we have

**Proposition 1.5** *If $p(N) \in \mathcal{A}(N)$ is an idempotent, then we can find an idempotent $p = \sum \alpha_i b_i \in \mathcal{A}$ such that $p(N) = \sum \alpha_i(N) b_i(N)$. Moreover, if the idempotent $p(N)$ is equivalent to the idempotent $q(N)$ in $\mathcal{A}(N)$, then so are the corresponding idempotents $p$ and $q$ in $\mathcal{A}$.*

**Corollary 1.6** *There exists an inclusion of $Gr(\mathcal{C}(N))$ into $Gr(O(\infty))$*

## 1.4 Quotients

Let $Gr(O(\infty))_\pm$ be the $\mathbb{Z}$-span of $Gr(O(\infty))$, which obviously is an abelian ring; similarly the representation ring $Gr(G)_\pm$ of the group $G$, $G = O(N)$ or $G = Sp(N/2)$ is defined to be the $\mathbb{Z}$-span of $Gr(G)$. Let $I_N$ be the set of Young diagrams defined in Def. 1.1 in connection with the group $G = O(N)$ or $G = Sp(N/2)$. We define $\mathcal{I}_N$ to be the ideal in $Gr(O(\infty))_\pm$ generated by the elements $[\lambda]$ with $\lambda \in I_N$.

**Lemma 1.7** *The ring $Gr(O(\infty))_\pm$ is the $\mathbb{Z}$-span of $\{[\lambda],\ \lambda \in D_N\} \cup \mathcal{I}_N$.*

*Proof*. Let us first observe that given a Young diagram $\lambda$ it suffices to show that $[\lambda]$ is congruent to some $\mathbb{Z}$-linear combination $\sum m_\nu[\nu]$ of diagrams $\nu \in D_N$ modulo $\mathcal{I}_N$. We shall prove this property in the symplectic case first. This will be done by induction with respect to alphabetical order for Young diagrams. There is nothing to show if the number $\lambda_1$ of columns of $\lambda$ is $\leq N+1$. Let now $\lambda$ be a Young diagram with $N+2$ columns and with $k$ boxes in its last column, and assume that the claim has already been shown for all smaller diagrams. Let $\tilde{\lambda}$ be the diagram $\lambda$ without the last column. Then $[\tilde{\lambda}] \in \mathcal{I}_N$, and hence also $[\tilde{\lambda}][1^k]$. But on the other hand, by Lemma 1.4 we also have

$$[\tilde{\lambda}][1^k] = \lambda + [\text{lower diagrams}]. \qquad (*)$$

Applying the induction assumption to the lower diagrams and solving for $\lambda$ we obtain the claim for $\lambda$.

If $\lambda$ has more than $N+2$ columns, we apply the same strategy. Let again $\tilde{\lambda}$ be the Young diagram $\lambda$ without its last column. By induction assumption $\tilde{\lambda}$ is equivalent to a linear combination $\sum m_\nu[\nu]$ of Young diagrams $\nu$ in $D_N \cup \mathcal{I}_N$ with $\nu < \tilde{\alpha}$. Using this, we obtain

$$[\lambda] + [\text{lower diagrams}] = [\tilde{\lambda}][1^k] = \sum_\nu m_\nu[\nu][1^k],$$

where the right hand side is a linear combination of elements in $\{[\nu],\ \nu \in D_N\} \cup \mathcal{I}_N$ by the results of the previous paragraph. Solving for $\lambda$ and using the induction assumption gives the desired claim.

To prove the claim in the orthogonal case, it will be more convenient to do the induction by applying the alphabetical order to the transposed Young diagrams. By definition of $\mathcal{I}_N$, there is nothing to show if $\lambda$ has only one row. If $\lambda = [n,k]$, the claim is shown by induction on $k$, using that $[[n]][[k]]$ is a linear combination of $[[n,k]]$ and diagrams smaller than $[n,k]$. If $\lambda$ has $r > 2$ rows, we define $\tilde{\lambda}$ to be the diagram $\lambda$ without the last row. As before, we can assume by induction assumption that $\tilde{\lambda} = \sum_\nu m_\nu[\nu]$ with $\nu \in D_N \cup I_N$ for all $\nu$ for which $m_\nu \neq 0$. We can now finish the proof as before in the symplectic case, using Lemma 1.4,(b).

**Proposition 1.8** *Let notations be as in the previous lemma. Then* $Gr(O(\infty))_\pm \cong Gr(\mathcal{C}(N))_\pm$ *and there exists a ring homomorphism* $\Psi : Gr(O(\infty))_\pm \to Gr(G)_\pm$ *which induces a bijection between the* $\mathbb{N}$*-span of* $\{[\lambda],\ \lambda \in D_N\}$ *and* $Gr(G)$ *and an isomorphism* $Grin_\pm/\mathcal{I}'_N \cong Gr(G)_\pm$.

*Proof.* It follows from Cor 1.6 that there exists an inclusion of the Grothendieck semiring $Gr(\mathcal{C}(N))$ defined in connection with the algebras $\mathcal{C}_n(N)$ into $Gr(O(\infty))$. In view of Prop. 1.2 the elements $[\lambda]$ with $\lambda \in D_n \cup I_N$ are contained in the image of this inclusion. It can be shown by induction with respect to alphabetical order, exactly as it was done in the previous lemma that for each Young diagram $\lambda$ the corresponding element $[\lambda] \in Gr(O(\infty))$ is in the $\mathbb{Z}$-span of the image of $Gr(\mathcal{C}(N))$ in $Gr(O(\infty))$; indeed, it suffices to solve for $[\lambda]$ in Eq (*) of the proof of Lemma 1.7 and apply the induction assumption for the lower diagrams.... This shows that $Gr(O(\infty))_\pm$ and $Gr(\mathcal{C}(N))_\pm$ are isomorphic.

The map $\Phi$ from $\mathcal{C}_n(N)$ onto $\mathrm{End}_G(V^{\otimes n})$ induces a map $\hat{\Phi}$ from $Gr(\mathcal{C}(N))$ onto the representation-semiring $Gr(G)$. Extending this map to $Gr(\mathcal{C}(N))_\pm$ and combining it with the isomorphism established in the last paragraph, we obtain a map $\Psi$ from $Gr(O(\infty))_\pm$ onto $Gr(G)_\pm$.

Next we want to show that $\mathcal{I}'_N$ is contained in the kernel of $\Psi$. By Prop. 1.2, there exists for each $\lambda \in I'_N$, a path idempotent $p_t \in \mathcal{C}_{n,\lambda}$ for which $p_t(N)$ is well-defined. The claim follows as soon as we can show that $\Phi(p_t(N)) = 0$. If $G = O(N)$ and $V$ is its $N$-dimensional vector representation, there exists a polynomial $P_\lambda$, due to El-Samra and King, such that $\dim p_t(N)V^{\otimes n} = P_\lambda(N)$ for any Young diagram $\lambda$ and any $N$-evaluable idempotent $p_t \in \mathcal{C}_{n,\lambda}$.

The factorization of these polynomials into linear factors is explicitly given from which one easily checks that $P_\lambda(N) = 0$ for $\lambda \in I'_N$ (see e.g. [W3], p. 186 for more details). The claim is similarly shown for $G = Sp(N/2)$, where we now evaluate the polynomials at $x = -N$, which determines the dimension of the corresponding $Sp(N/2)$ representation up to a sign.

By Lemma 1.7 the elements $[\lambda]$, $\lambda \in D'_N$ span the quotient $Gr(O(\infty))_\pm/\mathcal{I}'_N$. If $ker\tilde{\Phi}$ were strictly larger than $\mathcal{I}'_N$, Lemma 1.7 would imply that the set $\{\tilde{\Phi}([\lambda]),\ \lambda \in D_N\}$ was linearly dependent. This contradicts the fact that the irreducible representations of $G$ are labelled by $D'_N$.

# 2 Reduction modulo $\mathcal{I}_N$

## 2.1 Reflection groups

Let $W = W^{(m)}$ be the reflection group generated by $S_m$, with the obvious action on $\mathbb{R}^m$, and the affine reflection $s_o$ in the plane $x_1 + x_2 = \ell$ in the orthogonal case, and in the plane $x_1 = \ell$ in the symplectic case. With these natural actions, we also define a second action on $\mathbb{R}^m$ by $w.\lambda = w(\lambda + \rho^{(m)}) - \rho^{(m)}$. Observe that this also defines an action on Young diagrams, whose image is an element in $\mathbb{Z}^m$.

We observe that if $\lambda$ is a Young diagram with $\leq m$ rows, and $w \in W^{(m)}$ such that $w.\lambda$ is again a Young diagram, then its image does not depend on the choice of $m$. To see this, let $l_i = \lambda_i + \rho_i^{(m)}$. Then $s_o(l_1, l_2, l_3,\ ...) = (\ell - l_2, \ell - l_1, l_3,\ ...)$ in the orthogonal case, and $s_o(l_1, l_2,\ ...) = (\ell - l_1, l_2,\ ...)$ in the symplectic case. It is then straightforward to check that $w.\lambda$ is independent of $m$ for $w \in S_n$, and for $w = s_o$.

In view of these observations it will be convenient to embed all groups $W^{(m)}$ into a large group $W = W^{(\infty)}$. This group $W$ has Coxeter type $D_\infty$ for the $O(N)$-case, and type $B_\infty$ for the $Sp(N)$-cases. It is generated by the group $S_\infty$ of finite permutations on a countable set together with one additional reflection $s_o$. It can be conveniently described by a faithful representation on the space of sequences, on which $S_\infty$ acts via permutations of the entries of the sequence and where the reflection $s_0$ changes the sign of the first (for type $B_\infty$) or of the first two members of a sequence (for type $D_\infty$). In particular, we can define the sign $\varepsilon(w)$ of an element $w \in W$ to be $(-1)^m$, where $m$ is the number of factors if we write $w$ as a product of simple reflections.

Let $\rho$ be the sequence $(1 - N/2 - i)_{i\in\mathbb{N}}$ in the $O(N)$- case, and let $\rho$ be the sequence $(-N - i)_i$ in the $Sp(N)$-case. We also have an action of $W$ on the set of integer sequences $\lambda$ with only finitely many nonzero entries defined by $w.\lambda = w(\lambda + \rho) - \rho$. It is easy to see that for $\lambda$ a Young diagram this action coincides with the one defined in the last section.

**Lemma 2.1** *Let $\lambda \in \Lambda$ be a Young diagram with $\leq m$ rows.*

*(a) Let $w \in W^{(m)} \subset W$. Then $w.\lambda$ is well-defined independent of the choice of $m \in \mathbb{N} \cup \{\infty\}$.*

*(b)There exists an element $w \in W$ such that either $w.\lambda \in D_N \cup I_N$ or $w(\lambda+\rho)$ has two identical coordinates. Moreover, if $w.\lambda \in D_N$, it is contained in $\lambda$, i.e. $(w.\lambda)_i \leq \lambda_i$ for all $i$.*

*Proof*. Part(a) has already been shown in this section. We will prove the statements of part (b) for the orthogonal case, with the proofs for the symplectic case being very similar. Let $l_i = \lambda_i + \rho_i$, for $i \in \mathbb{N}$. If $l_1 + l_2 \leq 0$, there is nothing to show. Otherwise, let $w_1$ be the permutation such that the elements of $w_1 s_o(\lambda+\rho)$ are decreasing. If they are strictly decreasing, it follows from the fact that $(\lambda+\rho)_i = \rho_i$ for all but finitely many indices $i$ that $w_1 s_o(\lambda+\rho) = \mu+\rho$ for some Young diagram $\mu$. Moreover, one checks from the definition of $w_1$ that $w_1 s_o(\lambda+\rho)_i \leq (\lambda+\rho)_i$ for all $i$, i.e. $\mu \subset \lambda$. Continuing this procedure, we will eventually either end up with a Young diagram in $D_N \cup I_N$, or $w(\lambda+\rho)$ will have two identical entries, as claimed in (a). $\square$

## 2.2 Multiplicative characters

We call any ring homomorphism $\chi : Gr(O(\infty))_\pm \to \mathbb{C}$ a *multiplicative character* of $Gr(O(\infty))$. Using the duality with $G = O(N)$ or $G = Sp(N)/2$, it is easy to construct examples of characters for $\cup\mathcal{C}_n(N)$: If $g \in G$, $p[N] \in \mathcal{C}_n(N)$ and $Tr$ is the usual trace on $V^{\otimes n}$, we define $\chi = \chi_g$ by $\chi([p[N]]) = Tr(g^{\otimes n}\Phi(p[n]))$. It is easy to check that this defines a multiplcative character on $Gr(\mathcal{C}(N))_\pm$, and hence also on $Gr(O(\infty))_\pm$ (see Prop. 1.8).

We say that a character of $Gr(O(\infty))$ annihilates $\mathcal{I}_N$ up to degree $m$ if $\chi([\lambda]) = 0$ for any element $[\lambda]$ in $\mathcal{I}_N$ for which the Young diagram $\lambda$ has at most $m$ rows.

To produce such characters, recall that if $g$ is an orthogonal matrix of rank $2m+1$, with eigenvalues 1 and $\alpha_i^{\pm 1}$, $i = 1, 2, \ldots m$, then its character for the representation labeled by the Young diagram $\lambda$ is given by

$$\det(\alpha_j^{(\lambda+\rho^{(m)})_i} - \alpha_j^{-(\lambda+\rho^{(m)})_i}) / \det(\alpha_j^{\rho_i^{(m)}} - \alpha_j^{-\rho_i^{(m)}}); \tag{2}$$

here $\rho_i^{(m)} = m + 1/2 - i$. If we set $\alpha_j = e^{2\pi\sqrt{-1}\epsilon_j/\ell}$, this formula becomes

$$\det(\sin(2(\lambda+\rho^{(m)})_i\epsilon_j\pi)/\ell) / \det(\sin(2\rho_i^{(m)}\epsilon_j\pi/\ell); \tag{3}$$

Similarly, the character for a matrix $g \in Sp(m)$ with eigenvalues $\alpha_i^{\pm 1}$, $i = 1, 2, \ldots m$ is given by the same formulas as in Eq. 2 and 3, but now with $\rho_i^{(m)} = m + 1 - i$ and without the factor 2 under the sines in Eq. 3. We shall consider the following characters, with $N \in \mathbb{N}$ fixed:

(a) $G = O(M)$, $M = 2m+1$: Let $\ell = N + M - 2$, and let $g = g(\epsilon)$ be an orthogonal $M \times M$ matrix with eigenvalues 1 and $e^{\pm 2\pi\sqrt{-1}\epsilon_j\pi/\ell}$, where either all $\epsilon_j$ are integers, or all $\epsilon_j$ are half-integers.

(b) $G = Sp(m)$: Let $N \in \mathbb{N}$, set $\ell = N + m + 1$ and let $g$ be a symplectic matrix with eigenvalues $e^{\pm 2\pi\sqrt{-1}\epsilon_j\pi/\ell}$, where now all $\epsilon_j$ are integers.

In view of the Weyl group symmetries, we can assume without restriction of generality that $\ell > \epsilon_1 > \epsilon_2 > \ ... \ > \epsilon_m \geq 0$. Let $E_\ell$ be the set of all $m$-tuples $\epsilon = (\epsilon_j)$ satisfying these conditions, and let $\chi_{g(\epsilon)}([\lambda])$ be the orthogonal resp. symplectic character of the matrix $g(\epsilon)$ in the representation $V_\lambda$. Then the set of vectors $\{\chi_\lambda = (\chi_{g(\epsilon)}([\lambda])),\ \lambda \in D_N\}$ is a subset of the column vectors of the so-called $S$-matrix appearing in connection with representations of affine Kac-Moody algebras related to orthogonal and symplectic Lie algebras (see [Kc] Section 13, or also e.g.[TW], Section 9 for an elementary description for this setting). This matrix is invertible, hence these vectors are linearly independent. As a consequence of this, we obtain the following useful statement:

Let $a = \sum_{\lambda \in D_N} m_\lambda[\lambda]$ and let $b = \sum_{\mu \in D_N} n_\lambda[\lambda]$ be linear combinations of elements $[\lambda]$ in $D_N$ for which each diagram $\lambda$ has at most $m$ rows. Then $\chi_g(a) = \chi_g(b)$ for all $\chi_g$ as in (b) if and only if $a = b$.

## 2.3 The quotients $Gr(O(\infty))_\pm/\mathcal{I}_N$ and $Gr(O(\infty))_\pm/\mathcal{I}'_N$

To formulate the main result of this section, we define a second action of $W$ on Young diagrams by $w'.\lambda = (w.\lambda')'$.

**Theorem 2.2** *Let $G = O(N)$ or $G = Sp(N/2)$, with the ideals $\mathcal{I}_N$ resp. $\mathcal{I}'_N$ defined accordingly.*

*(a) The quotient $Gr(O(\infty))_\pm/\mathcal{I}_N$ has a $\mathbb{Z}$ basis $\{[\mu], \mu \in D_N\}$. For an arbitrary Young diagram $\lambda$, we either have $[\lambda] \in \mathcal{I}_N$ or there exists $w \in W$ such that $w.\lambda \in D_N$ and $[\lambda] \equiv \varepsilon(w)[w.\lambda]$ mod $\mathcal{I}_N$.*

*(b) The quotient $Gr(O(\infty))_\pm/\mathcal{I}'_N$ has a $\mathbb{Z}$ basis $\{[\mu], \mu \in D'_N\}$. For an arbitrary Young diagram $\lambda$, we either have $[\lambda] \in \mathcal{I}'_N$ or there exists $w \in W$ such that $w'.\lambda \in D_N$ and $[\lambda] \equiv \varepsilon(w)[w.\lambda]$ mod $\mathcal{I}'_N$. In particular, the $\mathbb{N}$-span of the elements $[\mu]$ mod $\mathcal{I}'_N$, $\mu \in \mathcal{D}'_N$ forms a sub-semiring of $Gr(O(\infty))_\pm/\mathcal{I}'_N$ which is isomorphic to $Gr(G)$.*

*Proof*. The statements in (a) are equivalent to the first two sentences in statement (b) by Lemma 1.3. The first sentence in (a) and (b) hence follows from Prop. 1.8. To prove the second sentence, it suffices to show that $\chi([w.\lambda]) = \varepsilon(w)\chi([\lambda])$ for all characters $\chi = \chi_g$ as described above. Indeed, by the remark at the end of the last section, this forces $[w.\lambda] \cong \varepsilon(w)[\lambda]$ mod $\mathcal{I}_N$. If $w \in S_m \subset W$ the claim follows from the determinant formulas for $\chi_g$, see Eq 2 and 3. Moreover, $\sin(2\pi(\ell - l_2)\epsilon_j/\ell) = (-1)^{2\epsilon_j+1}\sin(2\pi l_2\epsilon_j/\ell)$. Hence replacing $\lambda$ by $s_o.\lambda$ results in interchanging the first two columns of the matrix whose determinant gives the character, with a possible sign change for both columns in the orthogonal case (see Section 2.1). It follows that $\chi([s_o.\lambda]) = -\chi([\lambda])$. The proof for the symplectic case is similar, but easier.

## 3 Restriction Coefficients

Let $p_\lambda$ be a minimal idempotent in $\mathbb{C}S_n$. Then it can be written as a sum of mutually commuting minimal idempotents in $\mathcal{C}_n$. Let $b_\mu^\lambda$ be the number of those idempotents which are in $\mathcal{C}_{n,\mu}$; equivalently, $b_\mu^\lambda$ is the trace of $p_\lambda$ in an irreducible $\mathcal{C}_{n,\mu}$-module. Again, these coefficients depend on whether we take the orthogonal or symplectic Brauer algebra, which will be made precise below.

The connection of these coefficients with representation theory of Lie groups is easily established as follows. Let $V$ be an $N$-dimensional vector space with $N > n$. By Schur duality, $F^\lambda = p_\lambda V^{\otimes n}$ is an irreducible $Gl(N)$-module. As $N > n$, we obtain a faithful representation of the Brauer algebra $\mathcal{C}_n(N)$ on $V^{\otimes n}$ such that its image is isomorphic to $\mathrm{End}_G(V^{\otimes n})$ for $G = O(N)$ or $G = Sp(N/2)$. Hence it follows that the coefficient $b_\mu^\lambda$ determines the multiplicity of the simple $G$-module $V_\mu$ in $F^\lambda$. If $N \leq n$, this multiplicity will be denoted by $b_\mu^\lambda(N)$. The coefficients $b_\mu^\lambda$ were computed by Littlewood as follows where part (c) is a simple consequence of parts (a) and (b) as well as Lemma 1.3:

**Theorem 3.1** *(Littlewood) If $G = O(N)$, we have $b_\mu^\lambda = \sum_\nu c_{\mu\nu}^\lambda$, where the summation goes over all Young diagrams which have an even number of boxes in each row, and $c_{\mu\nu}^\lambda$ is the multiplicity of the simple $Gl(N)$-module $F^\nu$ in $F^\lambda \otimes F^\mu$. In particular, $b_\mu^\lambda = 0$ unless $\mu \subset \lambda$, and $b_\lambda^\lambda = 1$.*

*(b) If $G = Sp(N)$, we have $b_\mu^\lambda = \sum_\nu c_{\mu\nu}^\lambda$, where the summation goes over all Young diagrams which have an even number of boxes in each column, and $c_{\mu\nu}^\lambda$ is as in (a).*

*(c) The orthogonal and symplectic restriction coefficients are related by $b_\mu^\lambda(O) = b_{\mu'}^{\lambda'}(Sp)$*

We can now express the restriction coefficients $b_\mu^\lambda(N)$ as follows

**Theorem 3.2** *Let $G = O(N)$ or $G = Sp(N/2)$, and let $W$ be the reflection group of type $D_\infty$ resp of type $B_\infty$ as defined in Section 2.1. Then the multiplicity $b_\mu^\lambda(N)$ of a simple $G$-module $V_\mu$ in a simple $Gl(N)$-module $F^\lambda$ is given by*

$$b_\lambda(N) = \sum_{w\in W} \varepsilon(w) b_{w'.\mu}^\lambda.$$

*Proof.* Let $g \in G$ and let $\chi_g$ resp. $\chi_g^{(N)}$ be the characters induced by it on $\cup\mathcal{C}_n$ resp $\cup\bar{\mathcal{C}}_n(N)$. Moreover, Let $p_\lambda \in \mathbb{C}S_n$ be a minimal idempotent. Then we have

$$\sum_{\mu\in D'_N} b_\mu^\lambda(N)\chi_g^{(N)}([\mu]) = \chi(p_\lambda) = \sum_\mu b_\mu^\lambda \chi_g([\mu]) = \sum_{\mu\in D'_N} \sum_w \varepsilon(w) b_{w.\mu}^\lambda \chi_g([\mu]).$$

As $\chi_g^{(N)}([\mu]) = \chi_g([\mu])$ for $\mu \in D'_N$ and as the $\chi_g$ separate the $\mu \in D'_N$, the claim follows.

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