Math 294: Homework 3, Problem 4.

Consider a stock price process $S = (S_0, S_1, S_2)$ defined on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$ by

$$S(\omega_1) = (8, 16, 32)$$

$$S(\omega_2) = (8, 16, 28)$$

$$S(\omega_3) = (8, 16, 10)$$

$$S(\omega_4) = (8, 4, 10)$$

$$S(\omega_5) = (8, 4, 4)$$

$$S(\omega_6) = (8, 4, 2)$$

Note, this is not a binomial model. However, you can draw a tree to describe the possible paths followed by the stock price process. You should do this and then answer the following questions.

(a) For $t = 0, 1, 2$, let $\mathcal{F}_t = \sigma\{S_s : 0 \leq s \leq t\}$, the σ-algebra generated by the stock price process up to time t. The entire collection of σ-algebras $\{\mathcal{F}_t, t = 0, 1, 2\}$ is called a filtration. Each $\mathcal{F}_t$ is generated by a partition $\mathcal{P}_t$ of Ω. Determine what $\mathcal{P}_t$ is for $t = 0, 1, 2$. Write down the list of sets that constitute $\mathcal{F}_t$, for $t = 0, 1.$.

(b) Let P be a probability measure defined on $(\Omega, \mathcal{F}_2)$ where $P(\{\omega_1\}) = 0.5$, $P(\{\omega_2\}) = 0.2$, $P(\{\omega_3\}) = 0.1$, $P(\{\omega_4\}) = 0.1$ $P(\{\omega_5\}) = 0.05$, $P(\{\omega_6\}) = 0.05$. Let E denote expectation under P. Determine the conditional expectations:

$$E[S_2|\mathcal{F}_1],$$

$$E[S_1|\mathcal{F}_0].$$

(c) Let $\tau_1 = \min\{t \geq 0 : S_t \geq 15\} \wedge 2$ (here the mininum of the empty set is $+\infty$). Write out what τ_1 is as a function on Ω. Prove that τ_1 is a stopping time.

(d) Let $\tau_2 = \max\{t \geq 0 : S_t > 4\}$ (here the maximum of the empty set is defined to be zero). Write out what τ_2 is as a function on Ω. Show that τ_2 is *not* a stopping time.