Stochastic Processing Networks

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Maurice Belz (1897-1975)
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Statistical Methods for the Process Industries (1973)
Stochastic Processing Networks: What, Why and How?

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Stochastic Processing Networks (cf. Harrison ‘00)

An activity consumes from certain classes, produces for certain (possibly different) classes, and uses certain servers.
Stochastic Processing Networks

**SPN Activities are Very General**

- Queueing network
- Flexible servers, alternate routing
- Simultaneous actions
Semiconductor Wafer Fab: P. R. Kumar
Multiclass Queueing Network
Call Center: First Direct (branchless retail banking)
Larreche et al., INSEAD ‘97 (see also Gans, Koole, Mandelbaum ‘93)
Differentiated Service Center
(Parallel server system, alternate routing)
NxN Input Queued Packet Switch: Prabhakar

Diagram of NxN Input Queued Packet Switch:

- $A_1(n)$
- $A_11(n)$
- $A_{ii}(n)$
- $A_{ii}(n)$
- $A_{NN}(n)$
- $A_{NN}(n)$
- $L_{11}(n)$
- $L_{NN}(n)$
- $D_1(n)$
- $D_N(n)$
- $N$
2x2 Input Queued Packet Switch
Data Network (Roberts and Massoulie, ‘00)

Link

Route
Simultaneous Resource Possession
Stochastic Processing Networks

- **APPLICATIONS**
  Complex manufacturing, telecommunications, computer systems, service networks

- **FEATURES**
  Multiclass, service discipline, alternate routing, complex feedback, heavily loaded

- **PERFORMANCE MEASURES**
  Queuelength, workload and server idletime
QUESTIONS

- STABILITY
- PERFORMANCE ANALYSIS (when heavily loaded)
- CONTROL (involves performance analysis for “good” controls)
A SIMPLE EXAMPLE:
SINGLE SERVER QUEUE
M/M/1 Queue

- Poisson arrivals at rate $\lambda$ (independent of service times)
- i.i.d. exponential service times mean $m$
- FIFO order of service, infinite buffer
M/M/1 Queue

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- Traffic intensity $\rho = \lambda m$
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- Queue length is a birth-death process (Markov)
- Positive recurrent (stable) iff $\rho < 1$
- Stationary distribution $\pi_i = \rho^i (1 - \rho)$, $i = 0, 1, 2, \ldots$
- Mean steady-state queue length $L = \rho / (1 - \rho)$
M/M/1 Queue

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- Positive recurrent (stable) iff $\rho < 1$
- Stationary distribution $\pi_i = \rho^i(1-\rho)$, $i = 0,1,2,...$
- Mean steady-state queue length $L = \rho/(1-\rho) = \lambda W$
M/GI/1 Queue

λ

m, σ^2

1
Mean steady-state queue length

\[ L = \rho + \frac{\rho^2 + \lambda^2 \sigma_s^2}{2(1 - \rho)} \]  
(Pollaczek-Khintchine)
GI/GI/1 Queue (+mild reg. assumptions)
GI/GI/1 Queue (+mild reg. assumptions)

\[(1 - \rho)L \approx \frac{\lambda^2(\sigma_a^2 + \sigma_s^2)}{2} \quad \text{for } \rho \approx 1\]

(Smith '53, Kingman, '61)
M/M/1 Queue
(Simulation of Dynamics)

\[ \rho = \lambda = 0.9524 \]
M/M/1 Queue
(Simulation of Dynamics)

\( \lambda \)

\( m \)

\( \rho = \lambda = 0.9524 \)
M/M/1 Queue

(Simulation of Dynamics)

\[ \rho = \lambda = 0.9524 \]
GI/GI/1 Queue (Dynamics)

$Q(t) =$ queue length at time $t$

Start system empty (for simplicity)

**Theorem (A. Borovkov ’67, Iglehart-Whitt ’70):** For $\rho \approx 1$,

$$(1 - \rho)Q(\cdot/(1 - \rho)^2) \approx Q^*(\cdot)$$

where $Q^*(\cdot)$ is a one-dimensional reflecting Brownian motion

with drift $-m^{-1}$ and variance parameter $\lambda^3 \sigma_a^2 + m^{-3} \sigma_s^2$
One-dimensional Reflecting Brownian Motion

\[ Q^*(t) = X^*(t) + Y^*(t) \]

\[ Y^*(t) = \sup \{-X^*(s) : 0 \leq s \leq t\} \]

\[ X^* = \text{Brownian motion} \]
APPROMOXIMATE DYNAMIC MODELS

- Most SPNs cannot be analyzed exactly
- Consider approximate models (valid under some scaling limit, e.g., heavily loaded, many sources, many servers, large networks)
- Two main classes of approximate models:
  - Fluid models (functional law of large numbers)
  - Diffusion models (functional central limit theorem)
Last 15 years: development of a theory for establishing stability and heavy traffic diffusion approximations for open multiclass queueing networks with non-idling head-of-the-line (HL) service disciplines.

**Head-of-the-line**: service allocated to a buffer goes to the job at the head-of-the-line (jobs within buffers are in FIFO order).
## PERSPECTIVE

<table>
<thead>
<tr>
<th>MQN</th>
<th>SPN</th>
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<tbody>
<tr>
<td><strong>Sufficient conditions for</strong> &lt;br&gt;stability and diffusion approximations</td>
<td>e.g., parallel server system, packet switch</td>
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<tr>
<td><strong>e.g., LIFO, Processor Sharing</strong> &lt;br&gt;(single station, PS: network stability)</td>
<td>e.g., Internet congestion control / bandwidth sharing model</td>
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MOTIVATING EXAMPLES

Stability
Performance
Control
Two-Station Priority Queueing Network
(Rybko-Stolyar ‘92)
Two-Station Priority Queueing Network
(Rybko-Stolyar ‘92)

- Poisson arrivals at rate 1 to buffers 1 and 3
- Exponential service times: $m_i$ mean rate of service for buffer $i$
- Preemptive resume priority: * denotes high priority classes

$\lambda_1 = 1$  
$\lambda_3 = 1$  
$m_1$  
$m_3$  
$m_2$  
$m_4$
Two-Station Priority Queueing Network
(Rybko-Stolyar ‘92)

- Poisson arrivals at rate $\lambda_1 = 1$ to buffers 1 and 3
- Exponential service times: $m_i$ mean rate of service for buffer $i$
- Preemptive resume priority: * denotes high priority classes
- Simulation: $m_1 = m_3 = 0.33, m_2 = m_4 = 0.66$
- Traffic intensities: $\rho_1 = m_1 + m_4 = 0.99, \rho_2 = m_2 + m_3 = 0.99$
Two-Station Priority Queueing Network
(Rybko-Stolyar ‘92)

--- Server 1 (sum of queues 1 & 4) --- Server 2 (sum of queues 2 & 3)
Parallel Server System

\[ \lambda_1 = 0.05, \quad \lambda_2 = 1.2, \quad \lambda_3 = 0.35 \]

\[ m_1 = 0.5, \quad m_2 = 1, \quad m_3 = 1, \quad m_4 = 2 \]
Parallel Server System

Simulation with static priority discipline:
server 1 gives priority to buffer 1, server 2 gives priority to buffer 2

Queuelengths for buffer 1 ---, buffer 2 ---, buffer 3 --- versus time
Parallel Server System

Simulation with dynamic priority discipline:
server 1 gives priority to buffer 1, server 2 gives priority to buffer 2, except when queue 2 goes below threshold of size 10

Queue lengths for buffer 1 ---, buffer 2 ---, buffer 3 --- versus time
MAIN TOPICS FOR REMAINING LECTURES

- Open Multiclass HL Queueing Networks: Stability and Performance
- Control of Stochastic Processing Networks: Some Theory and Examples