Exercise 1. (a) Broken stick theorem tells us (or by brutal calculation), the distance of $X_{(1)}$ and the endpoint $\theta - 1$, is $1/n + 1$. Therefore $\hat{\theta} = X_{(1)} + n/(n + 1)$ is an unbiased estimator.

(b) A better estimator could be $\tilde{\theta} = (X_{(1)} + X_{(n)})/2 + 1/2$. The unbiasedness comes from symmetry, and the symmetry also tells us $\text{var}(X_{(1)}) = \text{var}(X_{(n)})$. Therefore

$$\text{var}(\hat{\theta}) = \frac{1}{4}[\text{var}(X_{(1)}) + \text{var}(X_{(n)})] + \frac{1}{2}\text{cov}(X_{(1)}, X_{(n)}) \leq \text{var}(X_{(1)}) = \text{var}(\tilde{\theta}).$$

Notice that the equivalence only holds when $X_{(1)}$ and $X_{(n)}$ are almost surely linearly dependent, which obviously is not the case. So it is strictly better, thus $\hat{\theta}$ is not UMVU.

(c) (i) The result comes from Lehmann-Scheffe Theorem. (ii) It is not sufficient. Given $X_{(1)}$, the distribution of $X$ is still depending on $\theta$. The exact form of distribution of $X|X_{(1)}$ is with probability $1/n$ on $X_{(1)}$, and with probability $(n - 1)/n$, uniformly on $[X_{(1)}, \theta]$.

If one find it difficult showing this, it can be shown by given some certain numbers. For example if given $X_{(1)} = 1$, the density of conditional distribution of $X$ at 1.5 is positive when $\theta = 1.7$, and 0 when $\theta = 1.4$. So it is still depend on $\theta$. This violates the definition of sufficiency.

(d) Yes. No matter how one permutes the data, the value of the smallest one among the data stay fixed.

Exercise 2. (a) Using Method I, suppose $f(X)$ is unbiased for $e^{-3\lambda}$, and we have

$$\sum_t f(t) \frac{\lambda^t}{t!} e^{-\lambda} = e^{-3\lambda}$$

which gives

$$\sum_t f(t) \frac{\lambda^t}{t!} = e^{-2\lambda} = \sum_t \frac{(-2\lambda)^t}{t!}.$$

Comparing the coefficients gives $f(t) = (-2)^t$.

(b) It is weird since it takes negative values, while the parameter to be estimated is a positive value. The plug in ensures that the estimator is positive.

(c) The variance is finite if and only if the second moment is finite, which is $E(f(X)^2) = E(4^X) = E(e^{X\log 4}) = M(\log 4)$. Since the moment generating function for Poisson distribution is finite on the positive real line, the variance of this estimator is finite.

Exercise 3. One non-random estimator which converges to $\mu$ in probability and have infinite variance for all $n$, is $\bar{X} + 1/X_{(n)}$.

It is easy to see that the estimator has infinite variance, since $X_{(n)}$ has positive density at 0. By the same argument from a homework, $E(1/X_{(n)}^2) = \infty$.

Now we show that this estimator is consistent. Notice that $\bar{X} \to \mu$ in probability by Law of Large Numbers. And for any positive number $M$, we will have $P(X_{(n)} \leq M) \to 0$ as $n \to \infty$. This is because
\[ P(X_{(n)} \leq M) = F(M)^n \] where \( F \) is the c.d.f. of the Laplace distribution. Notice that \( F(M) < 1 \) for all \( M \), so the right hand side converges to 0 when \( n \to \infty \).

Now for any \( \varepsilon > 0 \),

\[ P(|\bar{X} + 1/X_{(n)} - \mu| \geq 2\varepsilon) \leq P(|\bar{X} - \mu| \geq \varepsilon) + P(|1/X_{(n)}| \geq \varepsilon) = P(|\bar{X} - \mu| \geq \varepsilon) + P(|X_{(n)}| \leq \varepsilon^{-1}) \]

The right hand side converges to 0 when \( n \to \infty \) by the previous argument, so it is consistent.