Exercise I. (a) For a single sample, the density function is \( p_\lambda(x) = \frac{\lambda^x}{x!} e^{-\lambda} \). Take a double derivative to its log and take negative expectations gives that \( i(\lambda) = 1/\lambda \). Therefore the information of the sample is \( ni(\lambda) = n/\lambda \).

(b) The CR bound is just the reverse of the Fisher information, which is \( \lambda/n \).

(c) Since \( var(X_1) = \lambda \) for Poisson variables, \( var(\bar{X}) = \lambda/n \), which hits the CR bound, and this shows that it is UMVU.

(d) Use the canonical form of the exponential family, we get \( T(x) = \sum X_i \) and \( \eta = \log(\lambda) \). Sufficiency is done by the factorization theorem, and completeness is proven by the fact that \( \log(\lambda) \) contains an open interval in \( \mathbb{R} \) when \( \lambda > 0 \).

(e) \( Y \) follows the zero-deflated Poisson distribution. To get its probability function, just divide the probability that the \( X_i \) is non-zero, to make the summation of the probabilities to 1.

\[
P(Y = y) = \frac{\lambda^y}{y!(1 - e^{-\lambda})} e^{-\lambda}
\]

Take a log, and take twice partial derivative to \( \lambda \), we have

\[
\frac{\partial^2 P}{\partial \lambda^2} = -\frac{y}{\lambda^2} + \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2}
\]

The expectation of \( Y \) can be calculated using the fact that \( E(Y)P(X > 0) = E(X) \), which yields \( E(Y) = \lambda/(1 - e^{-\lambda}) \). Plug in and we have

\[
i_Y(\lambda) = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})^2}
\]

(f) Rao-Blackwell theorem shows that \( E(\delta | T) \) has a variance no larger than \( \delta \) if \( T \) is sufficient, and \( \delta \) is unbiased. The Lehmann-Scheffe says that any unbiased estimator based on the complete and sufficient statistic is UMVU. Rao-Blackwell does not address the UMVU property as Lehmann-Scheffe does, but when \( T \) is also complete in Rao-Blackwell, the conditional expectation is indeed UMVUE.

(g) \( (X_1 - X_2)^2 \) is an unbiased estimator for \( 2\lambda \), and \( \sum_i X_i \) is complete and sufficient by (d). Therefore \( \delta(X) \) is UMVU for \( 2\lambda \). Notice that \( 2\bar{X} \) is also an UMVUE for \( 2\lambda \) by (c), therefore by uniqueness of UMVUE, \( \delta(X) = 2\bar{X} \).

(h) \( \delta(X) \) is conditioning on \( \bar{X} \) which is a sufficient statistic, thus is free of \( \lambda \) and it is a statistic.

(i) As is shown in (c), \( var(\delta(X)) = var(2\bar{X}) = 4var(\bar{X}) = 4\lambda/n \).

(j) No, the conditional tuples are not complete since \( E(\bar{X} - \sum_i(X_i - \bar{X})^2/(n - 1)) = 0 \) which ruins the completeness property.

(k) Yes it is a statistic due to the fact that it is a function of the tuple \( (\sum_i X_i, \sum_i(X_i - \bar{X})^2) \), and these tuple is a sufficient statistic. \( (\sum_i X_i \) is sufficient, therefore adding more information keeps sufficiency.)

(l) Due to the tower property:

\[
E(\epsilon(X)) = E(E((X_1 - X_2)^2 | \sum X_i, \sum_i (X_i - \bar{X})^2)) = E((X_1 - X_2)^2) = 2\lambda = E(\delta(X)).
\]
(m) First part, by the decomposition of the variance:

\[
\text{var}((X_1 - X_2)^2) = \text{var}(E((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2)) + E(\text{var}((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2))
\geq \text{var}(E((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2)) = \text{var}(\varepsilon(X))
\]

Second part is automatically true since $\delta(X)$ is UMVU. And both of them are unbiased.

**Exercise 2.**

(i) It is easy to see that $f_\theta(x) = f_0(x - \theta)$ which is the definition of location family.

(ii) By the expansion of the canonical form, \((\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3)\) is a sufficient statistic.

(iii) For location family the variance is fixed. Therefore denote $C$ as the variance of $X_1$ when $\theta = 0$, and construct

\[
g(\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3) = S^2 - C
\]

where $S^2$ is the sample variance. Clearly $E(g) = 0$ for all $\theta$ and $g \neq 0$. 

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