Exercise I. (a) For a single sample, the density function is \( p_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} \). Take a double derivative to its log and take negative expectations gives that \( i(\lambda) = \frac{1}{\lambda} \). Therefore the information of the sample is \( ni(\lambda) = \frac{n}{\lambda} \).

(b) The CR bound is just the reverse of the Fisher information, which is \( \frac{\lambda}{n} \).

(c) Since \( \text{var}(X_1) = \lambda \) for Poisson variables, \( \text{var}(\bar{X}) = \frac{\lambda}{n} \), which hits the CR bound, and this shows that it is UMVU.

(d) Use the canonical form of the exponential family, we get \( T(x) = \sum_i X_i \) and \( \eta = \log(\lambda) \). Sufficiency is done by the factorization theorem, and completeness is proven by the fact that \( \log(\lambda) \) contains an open interval in \( \mathbb{R} \) when \( \lambda > 0 \).

(e) \( Y \) follows the zero-deflated Poisson distribution. To get its probability function, just divide the probability that the \( X_i \) is non-zero, to make the summation of the probabilities to 1.

\[
P(Y = y) = \frac{\lambda^y}{y!(1 - e^{-\lambda})} e^{-\lambda}
\]

Take a log, and take twice partial derivative to \( \lambda \), we have

\[
\frac{\partial^2 P}{\partial \lambda^2} = -\frac{y}{\lambda^2} + \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2}
\]

The expectation of \( Y \) can be calculated using the fact that \( E(Y)P(X > 0) = E(X) \), which yields \( E(Y) = \lambda/(1 - e^{-\lambda}) \). Plug in and we have

\[
i_Y(\lambda) = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})^2}
\]

(f) Rao-Blackwell theorem shows that \( E(\delta|T) \) has a variance no larger than \( \delta \) if \( T \) is sufficient, and \( \delta \) is unbiased. The Lehmann-Scheffé says that any unbiased estimator based on the complete and sufficient statistic is UMVU. Rao-Blackwell does not address the UMVU property as Lehmann-Scheffé does, but when \( T \) is also complete in Rao-Blackwell, the conditional expectation is indeed UMVUE.

(g) \( (X_1 - X_2)^2 \) is an unbiased estimator for \( 2\lambda \), and \( \sum_i X_i \) is complete and sufficient by (d). Therefore \( \delta(X) \) is UMVU for \( 2\lambda \). Notice that \( 2\bar{X} \) is also an UMVUE for \( 2\lambda \) by (c), therefore by uniqueness of UMVUE, \( \delta(X) = 2\bar{X} \).

(h) \( \delta(X) \) is a function of \( \bar{X} \) by construction, thus is free of \( \lambda \) and it is a statistic.

(i) As is shown in (c), \( \text{var}(\delta(X)) = \text{var}(2\bar{X}) = 4\text{var}(\bar{X}) = 4\lambda/n \).

(j) No, the conditional tuples are not complete since \( E(\bar{X} - \sum_i (X_i - \bar{X})^2/(n-1)) = 0 \) which ruins the completeness property.

(k) Yes it is a statistic due to the fact that it is a function of the tuple \( (\sum_i X_i, \sum_i (X_i - \bar{X})^2) \), and these two are statistics.

(l) Due to the tower property:

\[
E(\varepsilon(X)) = E(E((X_1 - X_2)^2|\sum_i X_i, \sum_i (X_i - \bar{X})^2)) = E((X_1 - X_2)^2) = 4\lambda = E(\delta(X)).
\]
(m) First part, by the decomposition of the variance:

\[
\text{var}((X_1 - X_2)^2) = \text{var}(E((X_1 - X_2)^2|\sum_i X_i, \sum_i (X_i - \bar{X})^2)) + E(\text{var}((X_1 - X_2)^2|\sum_i X_i, \sum_i (X_i - \bar{X})^2)) \\
\geq \text{var}(E((X_1 - X_2)^2|\sum_i X_i, \sum_i (X_i - \bar{X})^2)) = \text{var}(\varepsilon(X))
\]

Second part is automatically true since \(\delta(X)\) is UMVU. And both of them are unbiased.

**Exercise 2.** (i) It is easy to see that \(f_{\theta}(x) = f_0(x - \theta)\) which is the definition of location family.

(ii) By the expansion of the canonical form, \((\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3)\) is a sufficient statistic.

(iii) For location family the variance is fixed. Therefore denote \(C\) as the variance of \(X_1\) when \(\theta = 0\), and construct

\[
g(\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3) = S^2 - C
\]

where \(S^2\) is the sample variance. Clearly \(E(g) = 0\) for all \(\theta\) and \(g \neq 0\).