Name: __________________________
Student #: ______________________
TA’s Name: ______________________
Session #: ________________________

Instructions
1. NO CALCULATOR.
2. CLOSE BOOK, CLOSE NOTES.
3. ID WILL BE CHECKED. GET IT READY!
4. SHOW ALL YOUR WORK!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page 2</td>
<td>(31 points)</td>
</tr>
<tr>
<td>Page 3</td>
<td>(30 points)</td>
</tr>
<tr>
<td>Page 4</td>
<td>(22 points)</td>
</tr>
<tr>
<td>Page 5</td>
<td>(17 points)</td>
</tr>
<tr>
<td>Total</td>
<td>(100 points)</td>
</tr>
</tbody>
</table>
Questions 1 - 2 (8pts each) are multiple choices. There may be more than one correct answer. Circle all correct answers to get full credit. Incorrect responses will be penalized accordingly. (For example, if problem X has 2 correct answers and if Y chooses two correct ones and one incorrect (total circle three answers) then he will only receives 4 out of 8 since the incorrect one cancels the credit of one of the two correct choices.)

1. Which of the following assertion(s) is (are) correct? (a), (c)
   (a) \( \vec{F}(x, y) = (x^2 + y^2)\vec{i} - 2xy\vec{j} \) is not a gradient vector field.
   (b) \( \text{curl}(\vec{F}) \) is perpendicular to \( \vec{F} \).
   (c) \( \text{curl}(\nabla f) = 0 \) for any smooth function \( f(x, y, z) \).
   (d) \( \text{div}(\vec{F} \times \vec{G}) = \vec{F} \cdot \text{curl}(\vec{G}) - \vec{G} \cdot \text{curl}(\vec{F}) \).
   (e) None of them above.

2. Let \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \). Which of the following is (are) correct? (a), (b), (d)
   (a) \( \nabla^2 \left( \frac{1}{r} \right) = 0 \).
   (b) \( \text{curl}(\vec{r}) = 0 \).
   (c) \( \nabla \left( \frac{1}{r} \right) = 0 \).
   (d) \( \text{div}(\vec{r}) = 0 \).
   (e) None of them above.

For problems 3-7, you have to show all related steps. Only correct answer earns limited partial credits.

3. (15 pts) Find the second order Taylor approximation of \( f(x, y) = \sin(xy) + \cos(xy) \) at \( x_0 = 0, y_0 = 0 \).

   \( f(0, 0) = 1, f_x(0, 0) = 0, f_y(0, 0) = 0 \).
   \( f_{xx}(0, 0) = 0, f_{yy}(0, 0) = 0, f_{xy} = 1 \). Hence the Taylor approximation of 2nd order is

   \[
   f(h_1 + h_2) = 1 + \frac{1}{2} h_1 h_2 + E_2(f, \vec{h})
   \]
4. (15 pts) Let \( C = \{(x, \sqrt{1-x^2}) | 0 \leq x \leq \frac{1}{2}\} \). Find the length of this arc.

*Geometric method:* It is arc of degree \( \frac{\pi}{6} \). Hence the length is \( \frac{\pi}{6} \).

*Calculus method:* The curve can be parametrized as \( x = \cos \theta, y = \sin \theta \) for \( \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \).

By the formula

\[
L(C) = \int_{\pi/3}^{\pi/2} \sqrt{x'^2 + y'^2} \, d\theta
\]

it follows \( L(C) = \frac{\pi}{6} \).

5. (15 pts) Let \( \vec{F}(x, y, z) = \frac{\vec{y} - \vec{x}}{x^2 + y^2} \). Find the \( \text{curl}(\vec{F}) \).

*Direct calculation gives*

\[
\text{curl}(\vec{F}) = 0.
\]
6. (18 pts) Find the integral
\[
\int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy.
\]

*Exchange the order of the integral we have*
\[
\int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy = \int_0^2 \int_0^{2x} e^{x^2} \, dy \, dx = \int_0^2 2xe^{x^2} \, dx = \int_0^4 e^u \, du = e^4 - 1.
\]

7. Let \( E \) be the region bounded by \( y = x^2 + z^2 \) and \( y = 4 \). The following steps finds
\[
I = \int \int \int_E \sqrt{x^2 + z^2} \, dV.
\]

a) (4 pts) Express the integral in terms of the iterated integral in the following order
\[
\int_0^4 \int_{B(y)}^{C(x,y)} \int_{A(y)}^{D(x,y)} \sqrt{x^2 + z^2} \, dz \, dx \, dy.
\]

Find out the functions \( A(y), B(y), C(x, y), D(x, y) \).
\[
A(y) = -\sqrt{y}, B(y) = \sqrt{y}, C(x, y) = -\sqrt{y-x^2}, D(x, y) = \sqrt{y-x^2}
\]

(CONTINUE ON NEXT PAGE)
b) (10 pts) For any fixed $y$ between 0 and 4, express

$$\int_{A(y)}^{B(y)} \int_{C(x,y)}^{D(x,y)} \sqrt{x^2 + z^2} \, dz \, dx$$

as a double integral $\int_{\Omega(y)} \sqrt{x^2 + z^2} \, dA$.

Describe $\Omega(y)$ and find the double integral using the polar coordinate.

$\Omega(y)$ is a disk/ball of radius $\sqrt{y}$. Using the polar coordinate,

$$\int_{\Omega(y)} \sqrt{x^2 + z^2} \, dA = \int_0^{\sqrt{y}} \int_0^{2\pi} r^2 \, d\theta \, dr = \frac{2\pi}{3} y^{3/2}.$$

c) (7 pts) Using the result from the step b), find the triple integral $I$. You may use other method to compute $I$. But you shall only get the credit for part c) if you skip a) and b).

$$I = \int_0^4 \frac{2\pi}{3} y^{3/2} \, dy.$$  Direct computation gives that $I = \frac{128}{15} \pi$. 

END OF EXAM