Section 10.2. Summing an infinite series†

Example 1  Imagine that you want to go from a point $A$ toward a second point $B$ two miles away. Imagine that first you go one mile (Figure 1). Then you go a half mile further (Figure 2). Next you go half that distance (Figure 3), and so forth, so that at each stage you go half as far as you did in the previous stage. (a) How far have you gone after one, two, three, five, and eight stages? (b) Predict the limit of your distance from $A$ as the number of stages tends to $\infty$.

<table>
<thead>
<tr>
<th>Stages</th>
<th>Total distance</th>
<th>Decimal approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>1–2</td>
<td>$1 + \frac{1}{2}$</td>
<td>$1.5$</td>
</tr>
<tr>
<td>1–3</td>
<td>$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2$</td>
<td>$1.75$</td>
</tr>
<tr>
<td>1–5</td>
<td>$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$</td>
<td>$1.9375$</td>
</tr>
<tr>
<td>1–8</td>
<td>$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5$</td>
<td>$1.9921875$</td>
</tr>
</tbody>
</table>

**Answer:** (a) The distances are in the table above. (b) The distance seems to be approaching 2 miles.

**SOLUTION FOR INSTRUCTORS:**

(a) The distances are in the table above. (b) The distance seems to be approaching 2 miles.

†Lecture notes to accompany Section 10.2 of *Calculus, Early Transcendentals* by Rogawski. All or most of the examples in these notes are examples or exercises from Al Shenk’s calculus manuscript.
Example 2  
Now suppose you go one mile from A toward B in the first stage, \( \frac{1}{4} \) mile back toward A in the second stage, \( \left(\frac{1}{4}\right)^2 = \frac{1}{16} \) mile away from A in the third stage, \( \left(\frac{1}{4}\right)^3 = \frac{1}{64} \) mile toward A in the fourth stage, etc., so that at each stage you go one-fourth the distance and in the opposite direction from the previous stage.  
(a) How far have you gone after one, two, three, five, and eight stages?  
(b) Predict the limit of your distance from A as the number of stages tends to \( \infty \).

**Answer:**  
(a) The distances are in the table below.  
(b) The distance seems to be approaching 0.8 miles.

<table>
<thead>
<tr>
<th>Stages</th>
<th>Total distance</th>
<th>Decimal approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1–2</td>
<td>( 1 - \frac{1}{4} )</td>
<td>0.75</td>
</tr>
<tr>
<td>1–3</td>
<td>( 1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 )</td>
<td>0.8125</td>
</tr>
<tr>
<td>1–5</td>
<td>( 1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 )</td>
<td>0.79688</td>
</tr>
<tr>
<td>1–8</td>
<td>( 1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^5 ) ( + \left(\frac{1}{4}\right)^6 - \left(\frac{1}{4}\right)^7 )</td>
<td>0.7999987</td>
</tr>
</tbody>
</table>

SOLUTION FOR INSTRUCTORS:  
(a) The distances are in the table above.  
(b) The distance seems to be approaching 0.8 miles.

**Theorem 1 (Finite geometric series)**  
For any constant \( r \), and any nonnegative integer \( N \),

\[
\sum_{n=0}^{N} r^n = 1 + r + r^2 + r^3 + \cdots + r^N = \begin{cases} 
N + 1 & \text{if } r = 1 \\
\frac{1 - r^{N+1}}{1 - r} & \text{if } r \neq 1.
\end{cases}
\]

Example 3  
Give a concise formula for \( \sum_{n=2}^{511} (0.99)^n \) and find its approximate decimal value.

**Answer:**  
\[
\sum_{n=2}^{511} (0.99)^n = (0.99)^2 \left( \frac{1 - (0.99)^{510}}{1 - 0.99} \right) \approx 97.427602
\]

SOLUTION FOR INSTRUCTORS:

\[
\sum_{n=2}^{511} (0.99)^n = (0.99)^2 \sum_{n=0}^{509} (0.99)^n = (0.99)^2 \left( \frac{1 - (0.99)^{510}}{1 - 0.99} \right) \approx 97.427602
\]
Example 4  Evaluate \( \sum_{n=0}^{400} (-1)^n \).

Answer: \( \sum_{n=0}^{400} (-1)^n = 1 \)

SOLUTION FOR INSTRUCTORS:

\[
\sum_{n=0}^{400} (-1)^n = \left( \frac{1 - (-1)^{401}}{1 - (-1)} \right) = \frac{2}{2} = 1 \quad \text{\textbullet\quad Or:} \quad \sum_{n=0}^{400} (-1)^j = 1 - 1 + 1 - \cdots + 1 - 1 + 1 = 1
\]

Example 5  A rich uncle agrees to put $1 in a trust fund for you on your first birthday, $2 on your second birthday, $4 on your third birthday, and to put in twice the amount each year as the year before until you are eighteen. How much will be in the fund after your eighteenth birthday? \((2^{18} = 262,144)\)

Answer: After your eighteenth birthday, the fund will contain $263,143

SOLUTION FOR INSTRUCTORS:

After your eighteenth birthday, the fund will contain

\[
1 + 2 + 2^2 + 2^3 + \cdots + 2^{17} = \frac{1 - 2^{18}}{1 - 2}
\]

\[
= 2^{18} - 1 = 262,144 - 1 = 263,143 \text{ dollars.}
\]

Example 6  Suppose that when a ball is dropped from height \( h \) above a floor, it bounces up to a height of \( \frac{3}{4} h \) feet. What is the total distance the ball travels when it hits the ground for the tenth time if it is dropped from a height of 10 feet?

Answer: When the ball hits the ground for the tenth time, it has traveled \( 10 + 60[1 - (\frac{3}{4})^9] \approx 65.495 \text{ feet} \)

SOLUTION FOR INSTRUCTORS:

The ball falls 10 feet when it hits the ground the first time. • Then it bounces up \( \frac{3}{4}(10) \) (feet) and falls \( \frac{3}{4}(10) \) feet, so when it hits the ground a second time it has traveled a total of \( 10 + 2(\frac{3}{4})^2(10) \) feet. • On the next bounce it goes up and down \( (\frac{3}{4})^2(10) \) feet, and the total distance it has gone when it hits the ground the third time is \( 10 + 2(\frac{3}{4})^3(10) + 2(\frac{3}{4})^2(10) \) feet. • When it hits the ground for the tenth time, it has traveled

\[
10 + 2(\frac{3}{4})^2(10) + 2(\frac{3}{4})^3(10) + \cdots + 2(\frac{3}{4})^9(10) = 10 + 20\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^9
\]

\[
= 10 + 20\left(\frac{3}{4}\right)[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^8] = 10 + 15 \cdot \frac{1 - (\frac{3}{4})^9}{1 - \frac{3}{4}} = 10 + 60[1 - (\frac{3}{4})^9] \approx 65.495 \text{ feet}
\]

Theorem 2 (Infinite geometric series)  For \(|r| < 1\), the geometric series \( \sum_{n=0}^{\infty} r^n \) converges and has the value,

\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.
\]

The series diverges if \(|r| \geq 1\).
Example 7  Check the predictions (a) in Example 1 and (b) in Example 2.
Answer: (a) In Example 1 the limit of the distance is 2.  •  The prediction was correct.  (b) In Example 2 the limit of the distance is $\frac{4}{5}$.  •  The prediction was correct.

SOLUTION FOR INSTRUCTORS:

(a) In Example 1 the limit of the distance is $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 1$.  •  The prediction was correct.

(b) In Example 2 the limit of the distance is $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{1 - (-\frac{1}{4})} = \frac{4}{5}$.  •  The prediction was correct.

Example 8  Give the exact value of the infinite geometric series $\sum_{n=3}^{\infty} \left(-\frac{3}{4}\right)^n$.

Answer: $\sum_{n=3}^{\infty} \left(-\frac{3}{4}\right)^n = -\frac{21}{112}.$

SOLUTION FOR INSTRUCTORS:

$$\sum_{n=3}^{\infty} \left(-\frac{3}{4}\right)^n = (-\frac{3}{4})^3 + (-\frac{3}{4})^4 + (-\frac{3}{4})^5 + \cdots = (-\frac{3}{4})^3(1 + (-\frac{3}{4}) + (-\frac{3}{4})^2 + \cdots)$$

$$= (-\frac{3}{4})^3 \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n = (-\frac{3}{4})^3 \left(\frac{1}{1 - \left(-\frac{3}{4}\right)}\right) = -\frac{27}{4} = -\frac{21}{112}$$

Example 9  Express the infinite repeating decimal 0.002222... as a fraction.

Answer: 0.002222... = $\frac{1}{450}$

SOLUTION FOR INSTRUCTORS:

$$0.002222\cdots = \frac{2}{10^3} + \frac{2}{10^4} + \frac{2}{10^5} + \cdots = \frac{2}{10^3} \left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{1}{999} \left[1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right] = \frac{1}{999} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{999} \left(\frac{1}{1 - \frac{1}{10}}\right) = \left(\frac{1}{999}\right) \left(\frac{10}{9}\right) = \frac{1}{900}$$

A test for divergence

Theorem 3  If the numbers $\{a_n\}_{n=n_0}^{\infty}$ do not tend to zero as $n$ tends to $\infty$, then the infinite series $\sum_{n=n_0}^{\infty} a_n$ diverges.
Example 10  Show that the series $\sum_{n=1}^{\infty} \frac{2n+1}{n+3}$ diverges.

Answer: $a_n = \frac{2n+1}{n+3}$ tends to 2 and not to zero as $n \to \infty$. • The series diverges.

SOLUTION FOR INSTRUCTORS:

$$a_n = \frac{2n+1}{n+3} = \frac{2 + \frac{1}{n}}{1 + \frac{3}{n}}$$

tends to 2 and not to zero as $n \to \infty$. • The series diverges.

Interactive Examples

Work the following Interactive Examples on Shenk’s web page, http://www.math.ucsd.edu/˜asenk/\textsuperscript{†}:

Section 10.2: Examples 1–5

\textsuperscript{†}The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.