Section 10.3. Convergence of series with positive terms

**Theorem 1 (Convergence of series with positive terms)** An infinite series with positive terms either converges or diverges to $\infty$. The series converges if its partial sums are bounded and diverges if its partial sums are not bounded.

**Theorem 2 (The Integral Test for series with positive terms)** Suppose that the series

$$
\sum_{n=n_0}^{\infty} a_n
$$

with positive terms is such that $a_n = f(n)$ for integers $n \geq c$ with some $c$, where $y = f(x)$ is continuous on $[c, \infty)$ and decreasing for $x \geq c$. Then the infinite series (1) converges if and only if the improper integral

$$
\int_c^{\infty} f(x) \, dx
$$

converges.

Figures 1 and 2 show why the series converges (1) if the integral (2) converges. The area of the rectangles in Figure 1 is less than the area of the region in Figure 2. If the integral converges, then the area of the region in Figure 2 is bounded as $N \to \infty$, so that the partial sums of the series are bounded and the series converges.

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†Lecture notes to accompany Section 10.3 of *Calculus, Early Transcendentals* by Rogawski.
Figures 3 and 4 show why the series (1) diverges if the integral (2) diverges. The area of the rectangles in Figure 4 is greater than the area of the region in Figure 3. If the integral diverges, then the area of the region in Figure 3 tends to $\infty$ as $N \to \infty$, so that the partial sums of the series tend to $\infty$ and the series diverges.

\[ \text{Area} = \int_{c}^{N+1} f(x) \, dx \]

**FIGURE 3**

\[ \text{Area} = \sum_{n=c}^{N} a_n \]

**FIGURE 4**

**Example 1** Does $\sum_{n=1}^{\infty} ne^{-n^2}$ converge?

**Answer:** \( \int_{1}^{\infty} xe^{-x^2} \, dx = \frac{1}{2} e^{-1} \cdot \sum_{n=1}^{\infty} ne^{-n^2} \) converges by the Integral Test. (Some partial sums of the series are plotted in Figure A1.)

\[ s_N = \sum_{n=1}^{\infty} ne^{-n^2} \]

Figure A1
**Example 2** Does \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converge or diverge?

**Answer:**

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \infty \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n \ln n}
\]
diverges by the Integral Test. (Some partial sums of the series are plotted in Figure A2.)

\[
s_N = \sum_{n=2}^{N} \frac{1}{n \ln n}
\]

Figure A2

The Integral Test is used to establish the following result.

**Theorem 3 (Convergence of the p-series)** The infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots
\]

converges if \( p > 1 \) and diverges if \( p \leq 1 \).

**Example 3** Does \( \sum_{n=1}^{\infty} \frac{6}{n^{1.75}} \) converge or diverge and why?

**Answer:**

\[
\sum_{n=1}^{\infty} \frac{6}{n^{1.75}} = 6 \sum_{n=1}^{\infty} \frac{1}{n^{1.75}} \quad \text{converges because it is a constant multiplied by the p-series } \sum_{n=1}^{\infty} \frac{1}{n^{1.75}}
\]

with \( p = 1.75 > 1 \).

**Example 4** Does \( \sum_{n=2}^{\infty} \frac{1}{9 \sqrt{n}} \) converge or diverge and why?

**Answer:**

\[
\sum_{n=2}^{\infty} \frac{1}{9 \sqrt{n}} = \frac{1}{9} \sum_{n=2}^{\infty} \frac{1}{n^{1/2}} \quad \text{diverges because it is a constant multiplied by the p-series } \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}
\]

with \( p = \frac{1}{2} < 1 \).

**Example 5** Does \( \sum_{n=1}^{\infty} \frac{7n}{6n+1} \) converge or diverge and why?

**Answer:** The series diverges because \( a_n = \frac{7n}{6n+1} = \frac{7}{6 + \frac{1}{n}} \rightarrow \frac{7}{6} \) as \( n \rightarrow \infty \); the terms do not tend to 0.

**Example 6** The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is called the harmonic series. Does it converge or diverge?

**Answer:**

\[
\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges because it is the p-series with } p = 1.
\]
**Theorem 4 (The Comparison Test with positive terms)**

Suppose that $\sum_{n=n_0}^{\infty} a_n$ and $\sum_{n=n_0}^{\infty} b_n$ are series with positive terms.

(a) If $\sum_{n=n_0}^{\infty} b_n$ converges and there are constants $M$ and $N$ such that $a_n \leq M b_n$ for $n \geq N$, then $\sum_{n=n_0}^{\infty} a_n$ also converges.

(b) If $\sum_{n=n_0}^{\infty} b_n$ diverges and there are constants $M > 0$ and $N$ such that $a_n \geq M b_n$ for $n \geq N$, then $\sum_{n=n_0}^{\infty} a_n$ also diverges.

**Example 7**

Does the series $\sum_{n=0}^{\infty} \frac{(0.6)^n}{n+1}$ converge or diverge and why?

**Answer:** $\sum_{n=0}^{\infty} \frac{(0.6)^n}{n+1}$ converges by the Comparison Test with the convergent geometric series $\sum_{n=0}^{\infty} (0.6)^n$ because $\frac{(0.6)^n}{n+1} \leq (0.6)^n$ for $n \geq 0$. (The partial sums of the first series in Figure A7a are less than the partial sums of the second series in Figure A7b.)

**Example 8**

Does $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5n^3}$ converge or diverge and why?

**Answer:** $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5n^3}$ diverges by the Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ because $\frac{n^2 + 1}{5n^3} \geq \frac{n^2}{5n^3} = \frac{1}{5} \left( \frac{1}{n} \right)$ for $n \geq 1$, and $\frac{1}{5}$ is a positive constant.
**Theorem 5 (The Limit-Comparison Test with positive terms)** Suppose that

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L
\]

where the \(a_n\)'s and \(b_n\)'s are positive and \(L\) is either a nonnegative number or \(\infty\).

(a) If \(L\) is finite and positive, then \(\sum_{n=j_0}^{\infty} a_n\) converges if and only if \(\sum_{n=n_0}^{\infty} b_n\) converges.

(b) If \(L = 0\) and \(\sum_{n=n_0}^{\infty} b_n\) converges, then \(\sum_{n=n_0}^{\infty} a_n\) converges.

(c) If \(L = \infty\) and \(\sum_{n=n_0}^{\infty} b_n\) diverges, then \(\sum_{j=n_0}^{\infty} a_n\) also diverges.

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**Example 9**

Does \(\sum_{n=1}^{\infty} \frac{10(2^n)}{5^n - 1}\) converge or diverge and why?

**Answer:** \(\sum_{n=1}^{\infty} \frac{10(2^n)}{5^n - 1}\) converges by the Limit Comparison Test with the convergent geometric series \(\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n\) because

\[
\frac{5^n - 1}{2^n} = \frac{5^n}{2^n} \left(\frac{10(2^n)}{5^n - 1}\right) = \frac{10}{1 - 1/5^n} \to 10 \text{ as } n \to \infty,
\]

and 10 is a positive number.

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**Example 10**

Does the series \(\sum_{n=1}^{\infty} \frac{n}{6n^2 + 1}\) converge or diverge and why?

**Answer:** \(\sum_{n=1}^{\infty} \frac{n}{6n^2 + 1}\) diverges by the Limit Comparison Test with the divergent harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) because

\[
\frac{n}{6n^2 + 1} = \frac{n^2}{6n^2 + 1} = \frac{n^2}{n^2(6 + 1/n^2)} = \frac{1}{6 + 1/n^2} \to 1/6 \text{ as } n \to \infty,
\]

and \(1/6\) is a positive number.

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**Interactive Examples**

Work the following Interactive Examples on Shenk’s web page, [http://www.math.ucsd.edu/~ashenk/](http://www.math.ucsd.edu/~ashenk/):

Section 10.3: Examples 1–4

Section 10.4: Examples 1–4

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†The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.