Section 7.6. The method of partial fractions†

Example 1
(a) Show that \( \frac{1}{x^2 - x} = \frac{1}{x - 1} - \frac{1}{x} \).

(b) Use this equation to perform the integration \( \int \frac{1}{x^2 - x} \, dx \).

Answer: (a) Omitted: the answer is the solution. (b) \( \int \frac{1}{x^2 - x} \, dx = \ln |x - 1| - \ln |x| + C \)

Example 2
Find the partial-fraction decomposition of \( \frac{1}{x^2 - x} = \frac{1}{x(x - 1)} \) from Example 1.

Answer: \( \frac{1}{x(x - 1)} = \frac{1}{x} - \frac{1}{1 - x} \)

Example 3
Find the partial-fraction decomposition of \( y = \frac{3x^2 - 1}{x^3 - x} \).

Answer: \( \frac{3x^2 - 1}{x^3 - x} = \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \)

Example 4
Check the result of Example 3 by combining terms.

Answer: \( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} = \frac{3x^2 - 1}{x^3 - x} \)

Example 5
Use the result \( \frac{3x^2 - 1}{x^3 - x} = \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \) of Example 3 to perform the integration \( \int \frac{3x^2 - 1}{x^3 - x} \, dx \).

Answer: \( \int \frac{3x^2 - 1}{x^3 - x} \, dx = \ln |x| + \ln |x - 1| + \ln |x + 1| + C \)

Example 6
Find the antiderivative \( \int \frac{3x^3 + 12x + 1}{x^2 + 4} \, dx \).

Answer: Figure A6 \( \frac{3x^3 + 12x + 1}{x^2 + 4} = 3x + \frac{1}{x^2 + 4} \) \( \int \frac{3x^3 + 12x + 1}{x^2 + 4} \, dx = \frac{3}{2} x^2 + \frac{1}{2} \tan^{-1} \left( \frac{3x}{2} \right) + C \)

Figure A6

†Lecture notes to accompany Section 7.6 of Calculus, Early Transcendentals by Rogawski.
**Example 7**  Find the partial fraction decompostion of \( y = \frac{2x - 1}{(x - 1)^2} \).

Answer: \( \frac{2x - 1}{(x - 1)^2} = \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \)

**Example 8**  Use the result \( \frac{2x - 1}{(x - 1)^2} = \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \) from Example 7 to evaluate

\[
\int_{2}^{4} \frac{2x - 1}{(x - 1)^2} \, dx.
\]

Answer: \( \int_{2}^{4} \frac{2x - 1}{(x - 1)^2} \, dx = 2 \ln(3) + \frac{2}{3} \)

**Interactive Examples**

Work the following Interactive Examples on Shenk’s web page, \( \text{http://www.math.ucsd.edu/~ashenk/} \):¹

Section 8.4: Examples 1–4

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¹The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.