Math 20B. Lecture Examples.

Section 7.2. Integration by parts†

Theorem 1 (Integration by parts) At any value of $x$ where $u = u(x)$ and $v = v(x)$ have derivatives,

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx.$$

Rule 1 To find the integral $\int x^n e^{ax} \, dx$, $\int x^n \sin(ax) \, dx$ or $\int x^n \cos(ax) \, dx$ with a small positive integer $n$ and constant $a \neq 0$, integrate by parts $n$ times, differentiating the power of $x$ and integrating the exponential or trigonometric function each time.

Example 1  
(a) Find the antiderivative $\int x \cos x \, dx$. (b) Check the result with differentiation.

Answer: (a) $\int x \cos x \, dx = x \sin x + \cos x + C$  
(b) Product Rule: $\frac{d}{dx}(x \sin x + \cos x) = x \cos x$

Rule 2 (Mental integration by substitution) Suppose that $a$ is a nonzero constant.

To have $\frac{dv}{dx} = e^{ax}$, use $v = \frac{1}{a} e^{ax}$.

To have $\frac{dv}{dx} = \sin(ax)$, use $v = -\frac{1}{a} \cos(ax)$.

To have $\frac{dv}{dx} = \cos(ax)$, use $v = \frac{1}{a} \sin(ax)$.

Example 2 What is the area of the region between $y = xe^{-x}$ and the $x$-axis for $0 \leq x \leq 4$?

Answer: Figure A2  
.Area $= 1 - 5e^{-4}$

Figure A2

†Lecture notes to accompany Section 7.2 of Calculus, Early Transcendentals by Rogawski
Rule 3  To find an integral of
\[ y = \ln x, \quad y = \sin^{-1}x, \quad y = \cos^{-1}x, \quad y = \tan^{-1}x, \quad \text{or} \quad y = \cot^{-1}x, \]
multiplied by another function, first try the substitution,
\[ u = \ln x, \quad u = \sin^{-1}x, \quad u = \cos^{-1}x, \quad u = \tan^{-1}x, \quad \text{or} \quad u = \cot^{-1}x, \]
respectively. If this does not work, try integration by parts where the logarithm or inverse trigonometric function is differentiated. This may result in an integral that can be found by other means.

Example 3  (a) Find the indefinite integral \( \int x^3 \ln x \, dx \). (b) Check the result with differentiation.

Answer: (a) \( \int x^3 \ln x \, dx = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C \)  
(b) Product Rule: \( \frac{d}{dx} \left( \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 \right) = x^3 \ln x \)

Example 4  Evaluate \( \int_1^e \frac{\ln x}{x} \, dx \).

Answer: \( \int_1^e \frac{\ln x}{x} \, dx = \frac{1}{2} \)

Example 5  Find the antiderivatives of \( y = \tan^{-1}x \) \, dx.

Answer: \( \int \tan^{-1}x = x \tan^{-1}x - \frac{1}{2} \ln(x^2 + 1) + C \)

Interactive Examples
Work the following Interactive Examples on Shenk’s web page, http://www.math.ucsd.edu/~ashenk/.

Section 8.1: Examples 1–4

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\( ^{1} \)The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.