Section 7.6. The method of partial fractions†

Example 1 (a) Show that \( \frac{1}{x^2 - x} = \frac{1}{x - 1} - \frac{1}{x} \). (b) Use this equation to perform the integration \( \int \frac{1}{x^2 - x} \, dx \).

Answer: (a) Omitted: the answer is the solution. (b) \( \int \frac{1}{x^2 - x} \, dx = \ln |x - 1| - \ln |x| + C \)

Partial fractions with nonrepeated linear factors

A quotient \( p(x)/q(x) \) of polynomials \( y = p(x) \) and \( y = q(x) \) is proper if the degree of \( q(x) \) is less than the degree of \( p(x) \).

According to the Fundamental Theorem of Algebra, the denominator \( q(x) \) can in principle be factored into linear factors \( ax + b \) and irreducible quadratic factors \( ax^2 + bx + c \) with \( b^2 - 4ac < 0 \) which cannot be factored without using complex numbers.

We begin with the case of proper quotients where the denominator has only nonrepeated linear factors.

Rule 1 Suppose that the rational function \( \frac{p(x)}{q(x)} \) is proper and that the denominator has only nonrepeated linear factors, so that \( q(x) = (x - r_1)(x - r_2) \cdots (x - r_N) \) with distinct real numbers \( r_1, r_2, \ldots, r_N \). Then there are numbers \( A_1, A_2, \ldots, A_N \) such that

\[
\frac{p(x)}{q(x)} = \frac{p(x)}{(x - r_1)(x - r_2) \cdots (x - r_N)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_N}{x - r_N}.
\]

A procedure for finding the constants \( A_1, A_2, \ldots, A_N \) in equation (1) is described in the next examples.

Example 2 Use Rule 1 to find the partial-fraction decomposition of \( \frac{1}{x^2 - x} = \frac{1}{x(x - 1)} \) from Example 1.

Answer: \( \frac{1}{x(x - 1)} = \frac{1}{x} - \frac{1}{x - 1} \)

Example 3 Find the partial-fraction decomposition of \( y = \frac{3x^2 - 1}{x^3 - x} \).

Answer: \( \frac{3x^2 - 1}{x^3 - x} = \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \)

Example 4 Check the result of Example 3 by combining terms.

Answer: \( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} = \frac{3x^2 - 1}{x^3 - x} \)

†Lecture notes to accompany Section 7.6 of Calculus, Early Transcendentals by Rogawski.
Example 5  Use the result \( \frac{3x^2 - 1}{x^3 - x} = \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \) of Example 3 to perform the integration \( \int \frac{3x^2 - 1}{x^3 - x} \, dx \).

Answer: \( \int \frac{3x^2 - 1}{x^3 - x} \, dx = \ln |x| + \ln |x - 1| + \ln |x + 1| + C \)

Integrating improper quotients of polynomials
To integrate an improper quotient of polynomials, you need to write it first as the sum of a polynomial and a proper quotient. Then you integrate the resulting proper quotient directly or use partial fractions with it.

Example 6  Find the antiderivative \( \int \frac{3x^3 + 12x + 1}{x^2 + 4} \, dx \).

Answer: Figure A6  \[ \frac{3x^3 + 12x + 1}{x^2 + 4} = 3x + \frac{1}{x^2 + 4} \cdot \int \frac{3x^3 + 12x + 1}{x^2 + 4} \, dx = \frac{3x}{2} x^2 + \frac{1}{2} \tan^{-1} \left( \frac{1}{2} x \right) + C \]

Partial-fraction decompositions: the general case
The factorization of the denominator of a proper quotient of polynomials might include repeated linear or irreducible quadratic factors. Then the form of the partial-fraction decomposition is determined by the following rule, which includes the cases covered by Rule 1.

Rule 2  To form the partial-fraction decomposition of a proper quotient \( y = \frac{p(x)}{q(x)} \) of polynomials, include one or more terms for each factor of the denominator, according to the following prescriptions.

(a) for every nonrepeating linear factor \( x - r \), add a term of the form \( \frac{A}{x - r} \).

(b) If the linear factor \( x - r \) occurs to the kth power with \( k \geq 2 \), then add \( k \) terms of the form

\[ \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_k}{(x - r)^k} \]

(c) For each nonrepeating irreducible quadratic factor \( ax^2 + bx + c \), add a term

\[ \frac{Ax + B}{ax^2 + bx + c} \]

(d) If the irreducible quadratic factor \( ax^2 + bx + c \) occurs to the kth power with \( k \geq 2 \), add \( k \) terms of the form

\[ \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k} \]
Example 7  
Find the partial fraction decomposition of \( y = \frac{2x - 1}{(x - 1)^2} \).

Answer:
\[
\frac{2x - 1}{(x - 1)^2} = \frac{2}{x - 1} + \frac{1}{(x - 1)^2}
\]

Example 8  
Use the result \( \frac{2x - 1}{(x - 1)^2} = \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \) from Example 7 to evaluate

\[
\int_{\frac{3}{2}}^{4} \frac{2x - 1}{(x - 1)^2} \, dx.
\]

Answer:
\[
\int_{\frac{3}{2}}^{4} \frac{2x - 1}{(x - 1)^2} \, dx = 2 \ln(3) + \frac{2}{3}
\]

Interactive Examples

Work the following Interactive Examples on Shenk’s web page, http://www.math.ucsd.edu/~ashenk/:†

Section 8.4: Examples 1–4

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†The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.