24. Hölder Spaces

**Notation 24.1.** Let Ω be an open subset of \( \mathbb{R}^d \), \( BC(\Omega) \) and \( BC(\bar{\Omega}) \) be the bounded continuous functions on \( \Omega \) and \( \bar{\Omega} \) respectively. By identifying \( f \in BC(\bar{\Omega}) \) with \( f|_\Omega \in BC(\Omega) \), we will consider \( BC(\bar{\Omega}) \) as a subset of \( BC(\Omega) \). For \( u \in BC(\Omega) \) and \( 0 < \beta \leq 1 \) let

\[
\|u\| := \sup_{x \in \Omega} |u(x)| \quad \text{and} \quad [u]_\beta := \sup_{x,y \in \Omega} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\beta} \right\}.
\]

If \( [u]_\beta < \infty \), then \( u \) is Hölder continuous with holder exponent \(^{43}\) \( \beta \). The collection of \( \beta \)–Hölder continuous function on \( \Omega \) will be denoted by

\[
C^{0,\beta}(\Omega) := \{ u \in BC(\Omega) : [u]_\beta < \infty \}
\]

and for \( u \in C^{0,\beta}(\Omega) \) let

\[
\|u\|_{C^{0,\beta}(\Omega)} := \|u\|_u + [u]_\beta.
\]

**Remark 24.2.** If \( u : \Omega \to \mathbb{C} \) and \( [u]_\beta < \infty \) for some \( \beta > 1 \), then \( u \) is constant on each connected component of \( \Omega \). Indeed, if \( x \in \Omega \) and \( h \in \mathbb{R}^d \) then

\[
\left| \frac{u(x + th) - u(x)}{t} \right| \leq [u]_\beta t^{\beta}/t \to 0 \quad \text{as} \quad t \to 0
\]

which shows \( \partial_h u(x) = 0 \) for all \( x \in \Omega \). If \( y \in \Omega \) is in the same connected component as \( x \), then by Exercise 17.5 there exists a smooth curve \( \sigma : [0, 1] \to \Omega \) such that \( \sigma(0) = x \) and \( \sigma(1) = y \). So by the fundamental theorem of calculus and the chain rule,

\[
u(y) - u(x) = \int_0^1 \frac{d}{dt} u(\sigma(t)) dt = \int_0^1 0 \ dt = 0.
\]

This is why we do not talk about Hölder spaces with Hölder exponents larger than 1.

**Lemma 24.3.** Suppose \( u \in C^1(\Omega) \cap BC(\Omega) \) and \( \partial_i u \in BC(\Omega) \) for \( i = 1, 2, \ldots, d \), then \( u \in C^{0,1}(\Omega) \), i.e. \( [u]_1 < \infty \).

The proof of this lemma is left to the reader as Exercise 24.1.

**Theorem 24.4.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \). Then

1. Under the identification of \( u \in BC(\Omega) \) with \( u|_\Omega \in BC(\Omega) \), \( BC(\Omega) \) is a closed subspace of \( BC(\bar{\Omega}) \).
2. Every element \( u \in C^{0,\beta}(\Omega) \) has a unique extension to a continuous function (still denoted by \( u \)) on \( \bar{\Omega} \). Therefore we may identify \( C^{0,\beta}(\Omega) \) with \( C^{0,\beta}(\bar{\Omega}) \subset BC(\bar{\Omega}) \). (In particular we may consider \( C^{0,\beta}(\Omega) \) and \( C^{0,\beta}(\bar{\Omega}) \) to be the same when \( \beta > 0 \).)
3. The function \( u \in C^{0,\beta}(\Omega) \to \|u\|_{C^{0,\beta}(\Omega)} \in [0, \infty) \) is a norm on \( C^{0,\beta}(\Omega) \) which make \( C^{0,\beta}(\Omega) \) into a Banach space.

**Proof.** 1. The first item is trivial since for \( u \in BC(\Omega) \), the sup-norm of \( u \) on \( \Omega \) agrees with the sup-norm on \( \Omega \) and \( BC(\Omega) \) is complete in this norm.

\(^{43}\)If \( \beta = 1 \), \( u \) is is said to be Lipschitz continuous.
2. Suppose that \([u]_\beta < \infty\) and \(x_0 \in \partial \Omega\). Let \(\{x_n\}_{n=1}^\infty \subset \Omega\) be a sequence such that \(x_0 = \lim_{n \to \infty} x_n\). Then
\[
|u(x_n) - u(x_m)| \leq |u|_\beta |x_n - x_m|_\beta \to 0 \quad \text{as} \quad m, n \to \infty
\]
showing \(\{u(x_n)\}_{n=1}^\infty\) is Cauchy so that \(\bar{u}(x_0) := \lim_{n \to \infty} u(x_n)\) exists. If \(\{y_n\}_{n=1}^\infty \subset \Omega\) is another sequence converging to \(x_0\), then
\[
|u(x_n) - u(y_n)| \leq |u|_\beta |x_n - y_n|_\beta \to 0 \quad \text{as} \quad n \to \infty,
\]
showing \(\bar{u}(x_0)\) is well defined. In this way we define \(\bar{u}(x)\) for all \(x \in \partial \Omega\) and let \(\bar{u}(x) = u(x)\) for \(x \in \Omega\). Since a similar limiting argument shows
\[
|\bar{u}(x) - \bar{u}(y)| \leq |u|_\beta |x - y|_\beta
\]
for all \(x, y \in \bar{\Omega}\), it follows that \(\bar{u}\) is still continuous and \([\bar{u}]_\beta = [u]_\beta\). In the sequel we will abuse notation and simply denote \(\bar{u}\) by \(u\).

3. For \(u, v \in C^{0,\beta}(\Omega)\),
\[
[v + u]_\beta = \sup_{x, y \in \Omega, x \neq y} \left\{ \frac{|v(y) + u(y) - v(x) - u(x)|}{|x - y|_\beta} \right\}
\]
and for \(\lambda \in \mathbb{C}\) it is easily seen that \([\lambda u]_\beta = |\lambda|[u]_\beta\). This shows \([\cdot]_\beta\) is a semi-norm on \(C^{0,\beta}(\Omega)\) and therefore \(\| \cdot \|_{C^{0,\beta}(\Omega)}\) defined in Eq. (24.1) is a norm.

To see that \(C^{0,\beta}(\Omega)\) is complete, let \(\{u_n\}_{n=1}^\infty\) be a \(C^{0,\beta}(\Omega)\)-Cauchy sequence. Since \(BC(\Omega)\) is complete, there exists \(u \in BC(\Omega)\) such that \(\|u - u_n\|_u \to 0\) as \(n \to \infty\). For \(x, y \in \Omega\) with \(x \neq y\),
\[
\lim_{n \to \infty} \frac{|u(x) - u_n(x) - (u(y) - u_n(y))|}{|x - y|_\beta} = \lim_{n \to \infty} \frac{|(u_m - u_n)(x) - (u_m - u_n)(y)|}{|x - y|_\beta}
\]
and so we see that \(u \in C^{0,\beta}(\Omega)\). Similarly,
\[
\lim_{m \to \infty} \frac{|u(x) - u_n(x) - (u(y) - u_n(y))|}{|x - y|_\beta} \leq \lim_{m \to \infty} \frac{|(u_m - u_n)(x) - (u_m - u_n)(y)|}{|x - y|_\beta}
\]
and therefore \(\lim_{n \to \infty} \|u - u_n\|_{C^{0,\beta}(\Omega)} = 0\).

**Notation 24.5.** Since \(\Omega\) and \(\bar{\Omega}\) are locally compact Hausdorff spaces, we may define \(C_0(\Omega)\) and \(C_0(\bar{\Omega})\) as in Definition 10.29. We will also let
\[
C^{0,\beta}_0(\Omega) := C^{0,\beta}(\Omega) \cap C_0(\Omega) \quad \text{and} \quad C^{0,\beta}_0(\bar{\Omega}) := C^{0,\beta}(\bar{\Omega}) \cap C_0(\bar{\Omega}).
\]

It has already been shown in Proposition 10.30 that \(C_0(\Omega)\) and \(C_0(\bar{\Omega})\) are closed subspaces of \(BC(\Omega)\) and \(BC(\bar{\Omega})\) respectively. The next proposition describes the relation between \(C_0(\Omega)\) and \(C_0(\bar{\Omega})\).

**Proposition 24.6.** Each \(u \in C_0(\Omega)\) has a unique extension to a continuous function on \(\bar{\Omega}\) given by \(\bar{u} = u\) on \(\bar{\Omega}\) and \(\bar{u} = 0\) on \(\partial \Omega\) and the extension \(\bar{u}\) is in \(C_0(\bar{\Omega})\). Conversely if \(u \in C_0(\bar{\Omega})\) and \(u|_{\partial \Omega} = 0\), then \(u|_{\Omega} \in C_0(\Omega)\). In this way we may identify \(C_0(\Omega)\) with those \(u \in C_0(\bar{\Omega})\) such that \(u|_{\partial \Omega} = 0\).
Proof. Any extension $u \in C_0(\Omega)$ to an element $\bar{u} \in C(\bar{\Omega})$ is necessarily unique, since $\Omega$ is dense inside $\bar{\Omega}$. So define $\bar{u} = u$ on $\Omega$ and $\bar{u} = 0$ on $\partial\Omega$. We must show $\bar{u}$ is continuous on $\bar{\Omega}$ and $\bar{u} \in C_0(\bar{\Omega})$.

For the continuity assertion it is enough to show $\bar{u}$ is continuous at all points in $\partial\Omega$. For any $\epsilon > 0$, by assumption, the set $K_\epsilon := \{x \in \Omega : |u(x)| \geq \epsilon\}$ is a compact subset of $\Omega$. Since $\partial\Omega = \Omega \setminus \Omega$, $\partial\Omega \cap K_\epsilon = \emptyset$ and therefore the distance, $\delta := d(K_\epsilon, \partial\Omega)$, between $K_\epsilon$ and $\partial\Omega$ is positive. So if $x \in \partial\Omega$ and $y \in \Omega$ and $|y - x| < \delta$, then $|\bar{u}(x) - \bar{u}(y)| = |u(y)| < \epsilon$ which shows $\bar{u} : \Omega \rightarrow \mathbb{C}$ is continuous. This also shows $\{|\bar{u}| \geq \epsilon\} = \{|u| \geq \epsilon\} = K_\epsilon$ is compact in $\bar{\Omega}$ and hence also in $\Omega$. Since $\epsilon > 0$ was arbitrary, this shows $\bar{u} \in C_0(\bar{\Omega})$.

Conversely if $u \in C_0(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$ and $\epsilon > 0$, then $K_\epsilon := \{x \in \bar{\Omega} : |u(x)| \geq \epsilon\}$ is a compact subset of $\bar{\Omega}$ which is contained in $\Omega$ since $\partial\Omega \cap K_\epsilon = \emptyset$. Therefore $K_\epsilon$ is a compact subset of $\bar{\Omega}$ showing $u|_{\partial\Omega} \in C_0(\bar{\Omega})$.

Definition 24.7. Let $\Omega$ be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N} \cup \{0\}$ and $\beta \in (0, 1]$. Let $BC_k^k(\Omega)$ ($BC_k^k(\bar{\Omega})$) denote the set of $k$-times continuously differentiable functions $u$ on $\Omega$ such that $\partial^\alpha u \in BC^k(\Omega)$ ($\partial^\alpha u \in BC^k(\bar{\Omega})$) for all $|\alpha| \leq k$. Similarly, let $BC_k^{k, \beta}(\Omega)$ denote those $u \in BC_k^k(\Omega)$ such that $|\partial^\alpha u|_\beta < \infty$ for all $|\alpha| = k$. For $u \in BC_k^k(\Omega)$ let

$$
\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u \quad \text{and}
\|u\|_{C^{k, \beta}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u + \sum_{|\alpha| = k} |\partial^\alpha u|_\beta.
$$

Theorem 24.8. The spaces $BC_k^k(\Omega)$ and $BC_k^{k, \beta}(\Omega)$ equipped with $\|\cdot\|_{C^k(\Omega)}$ and $\|\cdot\|_{C^{k, \beta}(\Omega)}$ respectively are Banach spaces and $BC_k^k(\Omega)$ is a closed subspace of $BC_k^{k, \beta}(\Omega)$ and $BC_k^{k, \beta}(\Omega) \subset BC_k^k(\bar{\Omega})$. Also

$$
C_0^{k, \beta}(\Omega) = C_0^{k, \beta}(\bar{\Omega}) = \{u \in BC_k^{k, \beta}(\Omega) : \partial^\alpha u \in C_0(\Omega) \forall |\alpha| \leq k\}
$$

is a closed subspace of $BC_k^{k, \beta}(\Omega)$.

Proof. Suppose that $\{u_m\}_{m=1}^\infty \subset BC_k^k(\Omega)$ is a Cauchy sequence, then $\{\partial^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $BC^k(\Omega)$ for $|\alpha| \leq k$. Since $BC^k(\Omega)$ is complete, there exists $g_0 \in BC^k(\Omega)$ such that $\lim_{n \rightarrow \infty} \|\partial^\alpha u_n - g_0\|_u = 0$ for all $|\alpha| \leq k$. Letting $u := g_0$, we must show $u \in C^k(\Omega)$ and $\partial^\alpha u = g_0$ for all $|\alpha| \leq k$. This will be done by induction on $|\alpha|$. If $|\alpha| = 0$ there is nothing to prove. Suppose that we have verified $u \in C^l(\Omega)$ and $\partial^\alpha u = g_0$ for all $|\alpha| \leq l$ for some $l < k$. Then for $x \in \Omega$, $i \in \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}$ sufficiently small,

$$
\partial^\alpha u_n(x + te_i) = \partial^\alpha u_n(x) + \int_0^t \partial_{i} \partial^\alpha u_n(x + \tau e_i) d\tau.
$$

Letting $n \rightarrow \infty$ in this equation gives

$$
\partial^\alpha u(x + te_i) = \partial^\alpha u(x) + \int_0^t \partial_{i} \partial^\alpha u(x + \tau e_i) d\tau
$$

from which it follows that $\partial_{i} \partial^\alpha u(x)$ exists for all $x \in \Omega$ and $\partial_{i} \partial^\alpha u = g_{\alpha, i}$. This completes the induction argument and also the proof that $BC_k^k(\Omega)$ is complete.

\footnote{To say $\partial^\alpha u \in BC^k(\Omega)$ means that $\partial^\alpha u \in BC^k(\Omega)$ and $\partial^\alpha u$ extends to a continuous function on $\bar{\Omega}$.}
It is easy to check that $BC_k^k(\Omega)$ is a closed subspace of $BC^k(\Omega)$ and by using Exercise 24.1 and Theorem 24.4 that $BC_k^{k,\beta}(\Omega)$ is a subspace of $BC^k(\Omega)$. The fact that $C_0^{k,\beta}(\Omega)$ is a closed subspace of $BC_k^{k,\beta}(\Omega)$ is a consequence of Proposition 10.30.

To prove $BC_k^{k,\beta}(\Omega)$ is complete, let $\{u_n\}_{n=1}^{\infty} \subset BC_k^{k,\beta}(\Omega)$ be a Cauchy sequence. By the completeness of $BC^k(\Omega)$ just proved, there exists $u \in BC^k(\Omega)$ such that $\lim_{n \to \infty} \|u - u_n\|_{C^k(\Omega)} = 0$. An application of Theorem 24.4 then shows $\lim_{n \to \infty} \|\partial^\alpha u_n - \partial^\alpha u\|_{C^{0,\beta}(\Omega)} = 0$ for $|\alpha| = k$ and therefore $\lim_{n \to \infty} \|u - u_n\|_{C^{k,\beta}(\Omega)} = 0$. □

The reader is asked to supply the proof of the following lemma.

**Lemma 24.9.** The following inclusions hold. For any $\beta \in [0,1]$

$$BC^{k+1,0}(\Omega) \subset BC^{k,1}(\Omega) \subset BC^{k,\beta}(\Omega) \subset BC^{k+1,0}(\Omega).$$

**Definition 24.10.** Let $A : X \to Y$ be a bounded operator between two (separable) Banach spaces. Then $A$ is **compact** if $A[B_X(0,1)]$ is precompact in $Y$ or equivalently for any $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\|x_n\| \leq 1$ for all $n$ the sequence $y_n := Ax_n \in Y$ has a convergent subsequence.

**Example 24.11.** Let $X = \ell^2 = Y$ and $\lambda_n \in \mathbb{C}$ such that $\lim_{n \to \infty} \lambda_n = 0$, then $A : X \to Y$ defined by $(Ax)(n) = \lambda_n x(n)$ is compact.

**Proof.** Suppose $\{x_j\}_{j=1}^{\infty} \subset \ell^2$ such that $\|x_j\|^2 = \sum |x_j(n)|^2 \leq 1$ for all $j$. By Cantor’s Diagonalization argument, there exists $\{j_k\} \subset \{j\}$ such that, for each $n$, $\tilde{x}_k(n) = x_{j_k}(n)$ converges to some $\tilde{x}(n) \in \mathbb{C}$ as $k \to \infty$. Since for any $M < \infty$,

$$\sum_{n=1}^{\infty} |\tilde{x}(n)|^2 = \lim_{k \to \infty} \sum_{n=1}^{M} |\tilde{x}_k(n)|^2 \leq 1$$

we may conclude that $\sum_{n=1}^{\infty} |\tilde{x}(n)|^2 \leq 1$, i.e. $\tilde{x} \in \ell^2$.

Let $y_k := A\tilde{x}_k$ and $y := A\tilde{x}$. We will finish the verification of this example by showing $y_k \to y$ in $\ell^2$ as $k \to \infty$. Indeed if $\lambda_M^* = \max_{n \geq M} |\lambda_n|$, then

$$\|A\tilde{x}_k - A\tilde{x}\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2$$

$$= \sum_{n=1}^{M} |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \sum_{M+1}^{\infty} |\tilde{x}_k(n) - \tilde{x}(n)|^2$$

$$\leq \sum_{n=1}^{M} |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \|\tilde{x}_k - \tilde{x}\|^2$$

$$\leq \sum_{n=1}^{M} |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + 4|\lambda_M^*|^2\|\tilde{x}_k - \tilde{x}\|^2.$$ 

Passing to the limit in this inequality then implies

$$\lim_{k \to \infty} \|A\tilde{x}_k - A\tilde{x}\|^2 \leq 4|\lambda_M^*|^2 \to 0 \text{ as } M \to \infty.$$ □
Lemma 24.12. If $X \xrightarrow{A} Y \xrightarrow{B} Z$ are continuous operators such the either $A$ or $B$ is compact then the composition $BA : X \to Z$ is also compact.

Proof. If $A$ is compact and $B$ is bounded, then $BA(BX(0,1)) \subset B(ABX(0,1))$ which is compact since the image of compact sets under continuous maps are compact. Hence we conclude that $BA(BX(0,1))$ is compact, being the closed subset of the compact set $B(ABX(0,1))$.

If $A$ is continuous and $B$ is compact, then $A(BX(0,1))$ is a bounded set and so by the compactness of $B$, $BA(BX(0,1))$ is a precompact subset of $Z$, i.e. $BA$ is compact.

Proposition 24.13. Let $\Omega \subset \mathbb{R}^d$ such that $\overline{\Omega}$ is compact and $0 \leq \alpha < \beta \leq 1$. Then the inclusion map $i : C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ is compact.

Let $\{u_n\}_{n=1}^\infty \subset C^\beta(\overline{\Omega})$ such that $\|u_n\|_{C^\beta} \leq 1$, i.e. $\|u_n\|_\infty \leq 1$ and

$$|u_n(x) - u_n(y)| \leq |x - y|^\beta$$

for all $x, y \in \overline{\Omega}$. By Arzela-Ascoli, there exists a subsequence of $\{\tilde{u}_n\}_{n=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $u \in C^\alpha(\overline{\Omega})$ such that $\tilde{u}_n \to u$ in $C^\alpha$. Since

$$|u(x) - u(y)| \leq \lim_{n \to \infty} |\tilde{u}_n(x) - \tilde{u}_n(y)| \leq |x - y|^\beta,$$

$u \in C^\beta$ as well. Define $g_n := u - \tilde{u}_n \in C^\beta$, then

$$\|g_n\|_{C^\alpha} = \|u\|_{C^\alpha} \leq 2$$

and $g_n \to 0$ in $C^0$. To finish the proof we must show that $g_n \to 0$ in $C^\alpha$. Given $\delta > 0$,

$$[g_n]_\alpha = \sup_{x \neq y} \frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} \leq A_n + B_n$$

where

$$A_n = \sup \left\{ \frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} : x \neq y \text{ and } |x - y| \leq \delta \right\}$$

$$= \sup \left\{ \frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} : |x - y| \leq \delta \right\}$$

$$\leq \delta^\beta - \alpha \cdot [g_n]_\beta \leq 2\delta^\beta - \alpha$$

and

$$B_n = \sup \left\{ \frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} : |x - y| > \delta \right\} \leq 2\delta^{-\alpha} \|g_n\|_{C^0} \to 0 \text{ as } n \to \infty.$$  

Therefore,

$$\lim_{n \to \infty} \sup [g_n]_\alpha \leq \lim_{n \to \infty} A_n + \lim_{n \to \infty} B_n \leq 2\delta^\beta - \alpha + 0 \to 0 \text{ as } \delta \downarrow 0.$$  

This proposition generalizes to the following theorem which the reader is asked to prove in Exercise 24.2 below.

Theorem 24.14. Let $\Omega$ be a precompact open subset of $\mathbb{R}^d$, $\alpha, \beta \in [0,1]$ and $k, j \in \mathbb{N}_0$. If $j + \beta > k + \alpha$, then $C^{j,\beta}(\overline{\Omega})$ is compactly contained in $C^{k,\alpha}(\overline{\Omega})$. 

24.1. Exercises.

Exercise 24.1. Prove Lemma 24.3.

Exercise 24.2. Prove Theorem 24.14. Hint: First prove $C^{\beta, \beta}(\bar{\Omega}) \subset C^{\alpha, \alpha}(\bar{\Omega})$ is compact if $0 \leq \alpha < \beta \leq 1$. Then use Lemma 24.12 repeatedly to handle all of the other cases.