23. Sobolev Spaces

**Definition 23.1.** For \( p \in [1, \infty] \), \( k \in \mathbb{N} \) and \( \Omega \) an open subset of \( \mathbb{R}^d \), let

\[
W^{k,p}_{\text{loc}}(\Omega) := \{ f \in L^p(\Omega) : \partial^\alpha f \in L^p_{\text{loc}}(\Omega) \text{ (weakly) for all } |\alpha| \leq k \},
\]

\[
W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \text{ (weakly) for all } |\alpha| \leq k \},
\]

then the triangle inequality for the inner product \( \langle f, g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^\alpha f \cdot \overline{\partial^\alpha g} \, dm \) implies

\[
\| f + g \|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \| \partial^\alpha f + \partial^\alpha g \|_{L^p(\Omega)}^p \right)^{1/p} \leq \left( \sum_{|\alpha| \leq k} \left[ \| \partial^\alpha f \|_{L^p(\Omega)} + \| \partial^\alpha g \|_{L^p(\Omega)} \right]^p \right)^{1/p} \leq \left( \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{L^p(\Omega)}^p \right)^{1/p} + \left( \sum_{|\alpha| \leq k} \| \partial^\alpha g \|_{L^p(\Omega)}^p \right)^{1/p} = \| f \|_{W^{k,p}(\Omega)} + \| g \|_{W^{k,p}(\Omega)}.
\]

This shows \( \| \cdot \|_{W^{k,p}(\Omega)} \) defined in Eq. (23.1) is a norm. We now show completeness.

If \( \{ f_n \}_{n=1}^\infty \subset W^{k,p}(\Omega) \) is a Cauchy sequence, then \( \{ \partial^\alpha f_n \}_{n=1}^\infty \) is a Cauchy sequence in \( L^p(\Omega) \) for all \( |\alpha| \leq k \). By the completeness of \( L^p(\Omega) \), there exists \( g_\alpha \in L^p(\Omega) \) such that \( g_\alpha = L^p- \lim_{n \to \infty} \partial^\alpha f_n \) for all \( |\alpha| \leq k \). Therefore, for all \( \phi \in C_c^\infty(\Omega) \),

\[
\langle f, \partial^\alpha \phi \rangle = \lim_{n \to \infty} \langle f_n, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \lim_{n \to \infty} \langle \partial^\alpha f_n, \phi \rangle = (-1)^{|\alpha|} \lim_{n \to \infty} \langle g_\alpha, \phi \rangle.
\]

This shows \( \partial^\alpha f \) exists weakly and \( g_\alpha = \partial^\alpha f \) a.e. This shows \( f \in W^{k,p}(\Omega) \) and that \( f_n \to f \in W^{k,p}(\Omega) \) as \( n \to \infty \). ■
Example 23.3. Let \( u(x) := |x|^{-\alpha} \) for \( x \in \mathbb{R}^d \) and \( \alpha \in \mathbb{R} \). Then

\[
\int_{B(0,R)} |u(x)|^p \, dx = \sigma(S^{d-1}) \int_0^R \frac{1}{r^{\alpha p}} r^{d-1} \, dr = \sigma(S^{d-1}) \int_0^R r^{d-\alpha p-1} \, dr
\]

(23.4)

\[
= \sigma(S^{d-1}) \cdot \left\{ \begin{array}{ll}
\frac{R^{d-\alpha p}}{d-\alpha p} & \text{if } d-\alpha p > 0 \\
\infty & \text{otherwise}
\end{array} \right.
\]

and hence \( u \in L_{loc}^p(\mathbb{R}^d) \) iff \( \alpha < d/p \). Now \( \nabla u(x) = -\alpha |x|^{-\alpha-1} \hat{x} \) where \( \hat{x} := x/|x| \).

Hence if \( \nabla u(x) \) is to exist in \( L_{loc}^p(\mathbb{R}^d) \) it is given by \( -\alpha |x|^{-\alpha-1} \hat{x} \) which is in \( L_{loc}^p(\mathbb{R}^d) \) iff \( \alpha + 1 < d/p \), i.e. if \( \alpha < d/p - 1 = \frac{d-p}{p} \). Let us not check that \( u \in W^{1,p}_{loc}(\mathbb{R}^d) \) provided \( \alpha < d/p - 1 \). To do this suppose \( \phi \in C_c^\infty(\mathbb{R}^d) \) and \( \epsilon > 0 \), then

\[
-\langle u, \partial_i \phi \rangle = -\lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} u(x) \partial_i \phi(x) \, dx
\]

\[
= \lim_{\epsilon \downarrow 0} \left\{ \int_{|x| > \epsilon} \partial_i u(x) \phi(x) \, dx + \int_{|x| = \epsilon} u(x) \phi(x) \frac{x_i}{\epsilon} \, d\sigma(x) \right\}.
\]

Since

\[
\left| \int_{|x| = \epsilon} u(x) \phi(x) \frac{x_i}{\epsilon} \, d\sigma(x) \right| \leq \|\phi\|_\infty \sigma(S^{d-1}) \epsilon^{d-1-\alpha} \to 0 \quad \text{as } \epsilon \downarrow 0
\]

and \( \partial_i u(x) = -\alpha |x|^{-\alpha-1} \hat{x} \cdot e_i \) is locally integrable we conclude that

\[
-\langle u, \partial_i \phi \rangle = \int_{\mathbb{R}^d} \partial_i u(x) \phi(x) \, dx
\]

showing that the weak derivative \( \partial_i u \) exists and is given by the usual pointwise derivative.

23.1. Mollifications.

Proposition 23.4 (Mollification). Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( p \in [1, \infty) \) and \( u \in W^{k,p}_{loc}(\Omega) \). Then there exists \( u_n \in C_c^\infty(\Omega) \) such that \( u_n \to u \) in \( W^{k,p}_{loc}(\Omega) \).

Proof. Apply Proposition 19.12 with polynomials, \( p_\alpha(\xi) = \xi^\alpha \), for \( |\alpha| \leq k \).

Proposition 23.5. \( C_c^\infty(\mathbb{R}^d) \) is dense in \( W^{k,p}(\mathbb{R}^d) \) for all \( 1 \leq p < \infty \).

Proof. The proof is similar to the proof of Proposition 23.4 using Exercise 19.2 in place of Proposition 19.12.

Proposition 23.6. Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( p \geq 1 \), then

1. for any \( \alpha \) with \( |\alpha| \leq k \), \( \partial^\alpha : W^{k,p}(\Omega) \to W^{k-|\alpha|,p}(\Omega) \) is a contraction.
2. For any open subset \( V \subseteq \Omega \), the restriction map \( u \to u|_V \) is bounded from \( W^{k,p}(\Omega) \to W^{k,p}(V) \).
3. For any \( f \in C^k(\Omega) \) and \( u \in W^{k,p}_{loc}(\Omega) \), the \( fu \in W^{k,p}_{loc}(\Omega) \) and for \( |\alpha| \leq k \),

\[
\partial^\alpha (fu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} u,
\]

where \( \binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} \).
(4) For any \( f \in BC^k(\Omega) \) and \( u \in W^{k,p}_{loc}(\Omega) \), the \( fu \in W^{k,p}_{loc}(\Omega) \) and for \( |\alpha| \leq k \) Eq. (23.5) still holds. Moreover, the linear map \( u \in W^{k,p}(\Omega) \to fu \in W^{k,p}(\Omega) \) is a bounded operator.

**Proof.** Let \( \phi \in C_c^\infty(\Omega) \) and \( u \in W^{k,p}(\Omega) \), then for \( \beta \) with \( |\beta| \leq k - |\alpha| \),

\[
\langle \partial^\alpha u, \partial^\beta \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \partial^\beta \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^{\alpha+\beta} \phi \rangle = (-1)^{|\beta|} \langle \partial^{\alpha+\beta} u, \phi \rangle
\]

from which it follows that \( \partial^\beta (\partial^\alpha u) \) exists weakly and \( \partial^\beta (\partial^\alpha u) = \partial^{\alpha+\beta} u \). This shows that \( \partial^\alpha u \in W^{k-|\alpha|,p}(\Omega) \) and it should be clear that \( \| \partial^\alpha u \|_{W^{k-|\alpha|,p}(\Omega)} \leq \| u \|_{W^{k,p}(\Omega)} \).

Item 2. is trivial.

3 - 4. Given \( u \in W^{k,p}_{loc}(\Omega) \), by Proposition 23.4 there exists \( u_n \in C_c^\infty(\Omega) \) such that \( u_n \to u \) in \( W^{k,p}_{loc}(\Omega) \). From the results in Appendix A.1, \(fu_n \in C_c(\Omega) \subset W^{k,p}(\Omega) \) and

\[
\partial^\alpha (fu_n) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} u_n
\]

holds. Given \( V \subset \Omega \) such that \( \tilde{V} \) is compactly contained in \( \Omega \), we may use the above equation to find the estimate

\[
\| \partial^\alpha (fu_n) \|_{L^p(V)} \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \| \partial^\beta f \|_{L^\infty(V)} \| \partial^{\alpha-\beta} u_n \|_{L^p(V)} \leq C_\alpha(f,V) \sum_{\beta \leq \alpha} \| \partial^{\alpha-\beta} u_n \|_{L^p(V)} \leq C_\alpha(f,V) \| u_n \|_{W^{k,p}(\Omega)}
\]

wherein the last equality we have used Exercise 23.1 below. Summing this equation on \( |\alpha| \leq k \) shows

\[
\| fu_n \|_{W^{k,p}(\Omega)} \leq C(f,V) \| u_n \|_{W^{k,p}(\Omega)} \text{ for all } n
\]

where \( C(f,V) := \sum_{|\alpha| \leq k} C_\alpha(f,V) \). By replacing \( u_n \) by \( u_n - u_m \) in the above inequality it follows that \( \{ fu_n \}_{n=1}^\infty \) is convergent in \( W^{k,p}(\Omega) \) and since \( V \) was arbitrary \( fu_n \to fu \) in \( W^{k,p}_{loc}(\Omega) \). Moreover, we may pass to the limit in Eq. (23.6) and in Eq. (23.7) to see that Eq. (23.5) holds and that

\[
\| fu \|_{W^{k,p}(\Omega)} \leq C(f,V) \| u \|_{W^{k,p}(\Omega)} \leq C(f,V) \| u \|_{W^{k,p}(\Omega)}
\]

Moreover if \( f \in BC(\Omega) \) then constant \( C(f,V) \) may be chosen to be independent of \( V \) and therefore, if \( u \in W^{k,p}(\Omega) \) then \( fu \in W^{k,p}(\Omega) \).

**Alternative direct proof of 4.** We will prove this by induction on \( |\alpha| \). If \( \alpha = e_i \) then, using Lemma 19.9,

\[
-\langle fu, \partial_i \phi \rangle = -\langle u, f \partial_i \phi \rangle = -\langle u, \partial_i [f \phi] - \partial_i f \cdot \phi \rangle = \langle \partial_i u, f \phi \rangle + \langle u, \partial_i f \cdot \phi \rangle = \langle f \partial_i u + \partial_i f \cdot u, \phi \rangle
\]

showing \( \partial_i (fu) \) exists weakly and is equal to \( \partial_i (fu) = f \partial_i u + \partial_i f \cdot u \in L^p(\Omega) \).

Supposing the result has been proved for all \( \alpha \) such that \( |\alpha| \leq m \) with \( m \in [1,k) \). Let \( \gamma = \alpha + e_i \); with \( |\alpha| = m \), then by what we have just proved each summand in Eq. (23.5) satisfies \( \partial_i [\partial^\beta f \cdot \partial^{\alpha-\beta} u] \) exists weakly and

\[
\partial_i [\partial^\beta f \cdot \partial^{\alpha-\beta} u] = \partial^{\beta+e_i} f \cdot \partial^{\alpha-\beta} u + \partial^\beta f \cdot \partial^{\alpha-\beta+e_i} u \in L^p(\Omega).
\]
Therefore $\partial^\gamma (fu) = \partial_i \partial^\alpha (fu)$ exists weakly in $L^p(\Omega)$ and

$$\partial^\gamma (fu) = \sum_{\beta \leq \alpha} \left( \begin{array} {c} \alpha \\ \beta \end{array} \right) [\partial^{\beta+\epsilon} f \cdot \partial^{\alpha-\beta} u + \partial^{\beta} f \cdot \partial^{\alpha-\beta+\epsilon} u] = \sum_{\beta \leq \gamma} \left( \begin{array} {c} \gamma \\ \beta \end{array} \right) [\partial^{\beta} f \cdot \partial^{\gamma-\beta} u].$$

For the last equality see the combinatorics in Appendix A.1. 

**Theorem 23.7.** Let $\Omega$ be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

**Proof.** Let $\Omega_n := \{x \in \Omega : \text{dist}(x, \Omega) > 1/n\} \cap B(0, n)$, then

$$\bar{\Omega}_n \subset \{x \in \Omega : \text{dist}(x, \Omega) \geq 1/n\} \cap B(0, n) \subset \Omega_{n+1},$$

$\bar{\Omega}_n$ is compact for every $n$ and $\Omega_n \uparrow \Omega$ as $n \to \infty$. Let $V_0 = \Omega_3$, $V_j := \Omega_{j+3} \setminus \bar{\Omega}_j$ for $j \geq 1$, $K_0 := \Omega_2$ and $K_j := \Omega_{j+2} \setminus \Omega_{j+1}$ for $j \geq 1$ as in figure 41. Then $K_n \subset V_n$

![Figure 41. Decomposing $\Omega$ into compact pieces. The compact sets $K_0$, $K_1$ and $K_2$ are the shaded annular regions while $V_0$, $V_1$ and $V_2$ are the indicated open annular regions.](image)

for all $n$ and $\cup K_n = \Omega$. Choose $\phi_n \in C_c^\infty(V_n, [0, 1])$ such that $\phi_n = 1$ on $K_n$ and set $\psi_0 = \phi_0$ and

$$\psi_j = (1 - \psi_1 - \cdots - \psi_{j-1}) \phi_j = \phi_j \prod_{k=1}^{j-1} (1 - \phi_k)$$

for $j \geq 1$. Then $\psi_j \in C_c^\infty(V_n, [0, 1])$,

$$1 - \sum_{k=0}^{n} \psi_k = \prod_{k=1}^{n} (1 - \phi_k) \to 0 \text{ as } n \to \infty$$

so that $\sum_{k=0}^{\infty} \psi_k = 1$ on $\Omega$ with the sum being locally finite.

Let $\epsilon > 0$ be given. By Proposition 23.6, $u_n := \psi_n u \in W^{k,p}(\Omega)$ with $\text{supp}(u_n) \subset \subset V_n$. By Proposition 23.4, we may find $v_n \in C_c^\infty(V_n)$ such that
\[ \|u_n - v_n\|_{W^{k,p}(\Omega)} \leq \epsilon/2^{n+1} \text{ for all } n. \] Let \( v := \sum_{n=1}^{\infty} v_n \), then \( v \in C^\infty(\Omega) \) because the sum is locally finite. Since
\[ \sum_{n=0}^{\infty} \|u_n - v_n\|_{W^{k,p}(\Omega)} \leq \sum_{n=0}^{\infty} \epsilon/2^{n+1} = \epsilon < \infty, \]
the sum \( \sum_{n=0}^{\infty} (u_n - v_n) \) converges in \( W^{k,p}(\Omega) \). The sum, \( \sum_{n=0}^{\infty} (u_n - v_n) \), also converges pointwise to \( u - v \) and hence \( u - v = \sum_{n=0}^{\infty} (u_n - v_n) \) is in \( W^{k,p}(\Omega) \). Therefore \( v \in W^{k,p}(\Omega) \cap C^\infty(\Omega) \) and
\[ \|u - v\| \leq \sum_{n=0}^{\infty} \|u_n - v_n\|_{W^{k,p}(\Omega)} \leq \epsilon. \]

**Notation 23.8.** Given a closed subset \( F \subset \mathbb{R}^d \), let \( C^\infty(F) \) denote those \( u \in C(F) \) that extend to a \( C^\infty \) – function on an open neighborhood of \( F \).

**Remark 23.9.** It is easy to prove that \( u \in C^\infty(F) \) iff there exists \( U \in C^\infty(\mathbb{R}^d) \) such that \( u = U|_F \). Indeed, suppose \( \Omega \) is an open neighborhood of \( F \), \( f \in C^\infty(\Omega) \) and \( u = f|_F \in C^\infty(F) \). Using a partition of unity argument (making use of the open sets \( V_i \) constructed in the proof of Theorem 23.7), one may show there exists \( \phi \in C^\infty(\Omega, [0, 1]) \) such that \( \text{supp}(\phi) \subset \Omega \) and \( \phi = 1 \) on a neighborhood of \( F \). Then \( U := \phi f \) is the desired function.

**Theorem 23.10 (Density of \( W^{k,p}(\Omega) \cap C^\infty(\Omega) \) in \( W^{k,p}(\Omega) \)).** Let \( \Omega \subset \mathbb{R}^d \) be a manifold with \( C^0 \) – boundary, then for \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty) \), \( W^{k,p}(\Omega) \cap C^\infty(\Omega) \) is dense in \( W^{k,p}(\Omega) \). This may alternatively be stated by assuming \( \Omega \subset \mathbb{R}^d \) is an open set such that \( \bar{\Omega} = \Omega \) and \( \Omega \) is a manifold with \( C^0 \) – boundary, then \( W^{k,p}(\Omega) \cap C^\infty(\Omega) \) is dense in \( W^{k,p}(\Omega) \).

Before going into the proof, let us point out that some restriction on the boundary of \( \Omega \) is needed for assertion in Theorem 23.10 to be valid. For example, suppose
\[ \Omega_0 := \{ x \in \mathbb{R}^2 : 1 < |x| < 2 \} \quad \text{and} \quad \Omega := \Omega_0 \setminus \{(1, 2) \times \{0\} \} \]
and \( \theta : \Omega \to (0, 2\pi) \) is defined so that \( x_1 = |x| \cos \theta(x) \) and \( x_2 = |x| \sin \theta(x) \), see Figure 42. Then \( \theta \in BC^\infty(\Omega) \subset W^{k,\infty}(\Omega) \) for all \( k \in \mathbb{N}_0 \) yet \( \theta \) can not be

![Figure 42. The region \( \Omega_0 \) along with a vertical in \( \Omega \).](image-url)
approximated by functions from $C^\infty(\bar{\Omega}) \subset BC^\infty(\Omega_0)$ in $W^{1,p}(\Omega)$. Indeed, if this were possible, it would follow that $\theta \in W^{1,p}(\Omega_0)$. However, $\theta$ is not continuous (and hence not absolutely continuous) on the lines $\{x_1 = \rho\} \cap \Omega$ for all $\rho \in (1,2)$ and so by Theorem 19.30, $\theta \notin W^{1,p}(\Omega_0)$.

The following is a warm-up to the proof of Theorem 23.10.

**Proposition 23.11** (Warm-up). Let $f: \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuous function and $\Omega := \{x \in \mathbb{R}^d: x_d > f(x_1, \ldots, x_{d-1})\}$ and $C^\infty(\bar{\Omega})$ denote those $u \in C(\bar{\Omega})$ which are restrictions of $C^\infty$-functions defined on an open neighborhood of $\Omega$. Then for $p \in [1, \infty)$, $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

**Proof.** By Theorem 23.7, it suffices to show that any $u \in C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ may be approximated by elements of $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$. For $s > 0$ let $u_s(x) := u(x + se_d)$. Then it is easily seen that $\partial^\alpha u_s = (\partial^\alpha u)_s$ for all $\alpha$ and hence

$$u_s \in W^{k,p}(\Omega - se_d) \cap C^\infty(\Omega - se_d) \subset C^\infty(\bar{\Omega}) \cap W^{k,p}(\bar{\Omega}).$$

These observations along with the strong continuity of translations in $L^p$ (see Proposition 11.13), implies $\lim_{s \to 0} \|u - u_s\|_{W^{k,p}(\Omega)} = 0$. $\blacksquare$

### 23.1.1. Proof of Theorem 23.10

**Proof.** By Theorem 23.7, it suffices to show that any $u \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ may be approximated by elements of $C^\infty(\bar{\Omega}) \cap W^{k,p}(\bar{\Omega})$. To understand the main ideas of the proof, suppose that $\Omega$ is the triangular region in Figure 43 and suppose that we have used a partition of unity relative to the cover shown so that $u = u_1 + u_2 + u_3$ with $\text{supp}(u_i) \subset B_i$. Now concentrating on $u_1$ whose support is depicted as the grey shaded area in Figure 43. We now simply translate $u_1$ in the direction $v$ shown in Figure 43. That is for any small $s > 0$, let $w_s(x) := u_1(x + sv)$, then $w_s$ lives on the translated grey area as seen in Figure 43. The function $w_s$ extended to be zero off its domain of definition is an element of $C^\infty(\bar{\Omega})$ moreover it is easily seen, using the same methods as in the proof of Proposition 23.11, that $w_s \to u_1$ in $W^{k,p}(\Omega)$.

The formal proof follows along these same lines. To do this choose an at most countable locally finite cover $\{V_i\}_{i=0}^\infty$ of $\bar{\Omega}$ such that $\bar{V}_0 \subset \Omega$ and for each $i \geq 1$,
after making an affine change of coordinates, \( V_i = (-\epsilon, \epsilon)^d \) for some \( \epsilon > 0 \) and

\[
V_i \cap \Omega = \{(y, z) \in V_i : \epsilon > z > f_i(y)\}
\]

where \( f_i : (-\epsilon, \epsilon)^{d-1} \to (-\epsilon, \epsilon) \), see Figure 44 below. Let \( \{\eta_i\}_{i=0}^{\infty} \) be a partition of

![Figure 44](image)

**Figure 44.** The shaded area depicts the support of \( u_i = w\eta_i \).

unity subordinated to \( \{V_i\} \) and let \( u_i := w\eta_i \in C^\infty (V_i \cap \Omega) \). Given \( \delta > 0 \), we choose \( s \) so small that \( u_i(x) := u_i(x + se_d) \) (extended to be zero off its domain of definition) may be viewed as an element of \( C^\infty (\bar{\Omega}) \) and such that \( \|u_i - w_i\|_{W^{k,p}(\Omega)} < \delta/2^i \). For \( i = 0 \) we set \( w_0 := w_0 = w\eta_0 \). Then, since \( \{V_i\}_{i=1}^{\infty} \) is a locally finite cover of \( \Omega \), it follows that \( w := \sum_{i=0}^{\infty} u_i \in C^\infty (\bar{\Omega}) \) and further we have

\[
\sum_{i=0}^{\infty} \|u_i - w_i\|_{W^{k,p}(\Omega)} \leq \sum_{i=1}^{\infty} \delta/2^i = \delta.
\]

This shows

\[
u - w = \sum_{i=0}^{\infty} (u_i - w_i) \in W^{k,p} (\Omega)
\]

and \( \|u - w\|_{W^{k,p}(\Omega)} < \delta \). Hence \( w \in C^\infty (\bar{\Omega}) \cap W^{k,p} (\Omega) \) is a \( \delta \) – approximation of \( u \) and since \( \delta > 0 \) arbitrary the proof is complete.

**23.2. Difference quotients.** Recall from Notation 19.14 that for \( h \neq 0 \)

\[
\partial_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}.
\]

**Remark 23.12** (Adjoins of Finite Differences). For \( u \in L^p \) and \( g \in L^q \),

\[
\int_{\mathbb{R}^d} \partial_i^h u(x) g(x) \, dx = \int_{\mathbb{R}^d} \frac{u(x + he_i) - u(x)}{h} g(x) \, dx = - \int_{\mathbb{R}^d} u(x) \frac{g(x - he_i) - g(x)}{-h} \, dx
\]

\[
= - \int_{\mathbb{R}^d} u(x) \partial_i^{-h} g(x) \, dx.
\]

We summarize this identity by \( (\partial_i^h)^* = -\partial_i^{-h} \).

**Theorem 23.13.** Suppose \( k \in \mathbb{N}_0 \), \( \Omega \) is an open subset of \( \mathbb{R}^d \) and \( V \) is an open precompact subset of \( \Omega \).

1. If \( 1 \leq p < \infty \), \( u \in W^{k,p}(\Omega) \) and \( \partial_i u \in W^{k,p}(\Omega) \), then

\[
(23.8) \quad \|\partial_i^h u\|_{W^{k,p}(V)} \leq \|\partial_i u\|_{W^{k,p}(\Omega)}
\]

for all \( 0 < |h| < \frac{1}{2} \text{dist}(V, \Omega^c) \).
(2) Suppose that $1 < p \leq \infty$, $u \in W^{k,p} (\Omega)$ and assume there exists a constant $C(V) < \infty$ such that
\[
\| \partial_h^{k} u \|_{W^{k,p}(V)} \leq C(V) \text{ for all } 0 < |h| < \frac{1}{2} \text{dist}(V, \Omega').
\]
Then $\partial_{h} u \in W^{k,p}(V)$ and $\| \partial_{h} u \|_{W^{k,p}(V)} \leq C(V)$. Moreover if $C := \sup_{V \subset \subset \Omega} C(V) < \infty$ then in fact $\partial_{h} u \in W^{k,p}(\Omega)$ and there is a constant $c < \infty$ such that
\[
\| \partial_{h} u \|_{W^{k,p}(\Omega)} \leq c \left( C + \| u \|_{L^p(\Omega)} \right).
\]

**Proof.** 1. Let $|\alpha| \leq k$, then
\[
\| \partial^\alpha \partial_h^k u \|_{L^p(V)} = \| \partial_h^k \partial^\alpha u \|_{L^p(V)} \leq \| \partial_h \partial^\alpha u \|_{L^p(V)}
\]
wherein we have used Theorem 19.22 for the last inequality. Eq. (23.8) now easily follows.

2. If $\| \partial_h^k u \|_{W^{k,p}(V)} \leq C(V)$ then for all $|\alpha| \leq k$,
\[
\| \partial_h^k \partial^\alpha u \|_{L^p(V)} = \| \partial^\alpha \partial_h^k u \|_{L^p(V)} \leq C(V).
\]
So by Theorem 19.22, $\partial_{h} \partial^\alpha u \in L^p(V)$ and $\| \partial_{h} \partial^\alpha u \|_{L^p(V)} \leq C(V)$. From this we conclude that $\| \partial^\beta u \|_{L^p(V)} \leq C(V)$ for all $0 < |\beta| \leq k+1$ and hence $\| u \|_{W^{k+1,p}(V)} \leq c \left[ C(V) + \| u \|_{L^p(V)} \right]$ for some constant $c$. \[\]

**Notation 23.14.** Given a multi-index $\alpha$ and $h \neq 0$, let
\[
\partial_h^\alpha := \prod_{i=1}^{d} (\partial_{h_i})^{\alpha_i}.
\]

The following theorem is a generalization of Theorem 23.13.

**Theorem 23.15.** Suppose $k \in \mathbb{N}_0$, $\Omega$ is an open subset of $\mathbb{R}^d$, $V$ is an open precompact subset of $\Omega$ and $u \in W^{k,p}(\Omega)$.

(1) If $1 \leq p < \infty$ and $|\alpha| \leq k$, then $\| \partial_h^k u \|_{W^{k-|\alpha|}(V)} \leq \| u \|_{W^{k,p}(\Omega)}$ for $h$ small.

(2) If $1 < p \leq \infty$ and $\| \partial_h^k u \|_{W^{k,p}(V)} \leq C$ for all $|\alpha| \leq j$ and $h$ near 0, then $u \in W^{k+j,p}(V)$ and $\| \partial_h^k u \|_{W^{k,p}(V)} \leq C$ for all $|\alpha| \leq j$.

**Proof.** Since $\partial_h^\alpha = \prod_{i=1}^{d} \partial_{h_i}^{\alpha_i}$, item 1. follows from Item 1. of Theorem 23.13 and induction on $|\alpha|$.

For Item 2., suppose first that $k = 0$ so that $u \in L^p(\Omega)$ and $\| \partial_h^k u \|_{L^p(V)} \leq C$ for $|\alpha| \leq j$. Then by Proposition 19.16, there exists $\{h_l\}_{l=1}^{\infty} \subset \mathbb{R} \setminus \{0\}$ and $v \in L^p(V)$ such that $h_l \to 0$ and $\lim_{l \to \infty} \langle \partial_h^k u, \phi \rangle = \langle v, \phi \rangle$ for all $\phi \in C_0^\infty (V)$. Using Remark 23.12,
\[
\langle v, \phi \rangle = \lim_{l \to \infty} \langle \partial_h^k u, \phi \rangle = (-1)^{|\alpha|} \lim_{l \to \infty} \langle u, \partial_h^{-\alpha} \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle
\]
which shows $\partial^\alpha u = v \in L^p(V)$. Moreover, since weak convergence decreases norms,
\[
\| \partial^\alpha u \|_{L^p(V)} = \| v \|_{L^p(V)} \leq C.
\]

For the general case if $k \in \mathbb{N}$, $u \in W^{k,p}(\Omega)$ such that $\| \partial_h^k u \|_{W^{k,p}(V)} \leq C$, then (for $p \in (1, \infty)$, the case $p = \infty$ is similar and left to the reader)
\[
\sum_{|\beta| \leq k} \| \partial_h^k \partial^\beta u \|_{L^p(V)}^p = \sum_{|\beta| \leq k} \| \partial^\beta \partial_h^k u \|_{L^p(V)}^p = \| \partial_h^k u \|_{W^{k,p}(V)}^p \leq C_p.
\]
As above this implies $\partial^{\alpha} \partial^{\beta} u \in L^{p}(V)$ for all $|\alpha| \leq j$ and $|\beta| \leq k$ and that
$$\|\partial^{\alpha} u\|_{W^{k,p}(V)}^{p} = \sum_{|\beta| \leq k} \|\partial^{\alpha} \partial^{\beta} u\|_{L^{p}(V)}^{p} \leq C^{p}.$$

23.3. Sobolev Spaces on Compact Manifolds.

**Theorem 23.16** (Change of Variables). Suppose that $U$ and $V$ are open subsets of $\mathbb{R}^{d}$, $T \in C^{k}(U, V)$ be a $C^{k}$ - diffeomorphism such that $\|\partial^{\alpha} T\|_{BC(U)} < \infty$ for all $1 \leq |\alpha| \leq k$ and $\epsilon := \inf_{U} |\det T'| > 0$. Then the map $T^{*} : W^{k,p}(V) \to W^{k,p}(U)$ defined by $u \in W^{k,p}(V) \to T^{*} u := u \circ T \in W^{k,p}(U)$ is well defined and is bounded.

**Proof.** For $u \in W^{k,p}(V) \cap C^{\infty}(V)$, repeated use of the chain and product rule implies,

\[
(u \circ T)^{(l)} = (u^{(l)} \circ T) T^{l} + \sum_{j=1}^{l-1} \binom{l}{j} (u^{(j)} \circ T) T^{j} \circ T^{l-j} + (u^{(l)} \circ T) T^{l}
\]

This equation and the boundedness assumptions on $T^{(j)}$ for $1 \leq j \leq k$ implies there is a finite constant $K$ such that

\[
|(u \circ T)^{(l)}| \leq K \sum_{j=1}^{l} |u^{(j)} \circ T| \text{ for all } 1 \leq l \leq k.
\]

By Hölder’s inequality for sums we conclude there is a constant $K_{p}$ such that

\[
\sum_{|\alpha| \leq k} |\partial^{\alpha} (u \circ T)|^{p} \leq K_{p} \sum_{|\alpha| \leq k} |\partial^{\alpha} u|^{p} \circ T
\]

and therefore

\[
\|u \circ T\|_{W^{k,p}(U)}^{p} \leq K_{p} \sum_{|\alpha| \leq k} \int_{U} |\partial^{\alpha} u|^{p} (T(x)) \, dx.
\]

Making the change of variables, $y = T(x)$ and using

\[
dy = |\det T'(x)| \, dx \geq \epsilon dx,
\]

we find

\[
\|u \circ T\|_{W^{k,p}(U)}^{p} \leq K_{p} \sum_{|\alpha| \leq k} \int_{U} |\partial^{\alpha} u|^{p} (T(x)) \, dx
\]

\[
\leq \frac{K_{p}}{\epsilon} \sum_{|\alpha| \leq k} \int_{V} |\partial^{\alpha} u|^{p} (y) \, dy = \frac{K_{p}}{\epsilon} \|u\|_{W^{k,p}(V)}^{p}.
\]
This shows that \( T^* : W^{k,p}(V) \cap C^\infty (V) \to W^{k,p}(U) \cap C^\infty (U) \) is a bounded operator. For general \( u \in W^{k,p}(V) \), we may choose \( u_n \in W^{k,p}(V) \cap C^\infty (V) \) such that \( u_n \to u \) in \( W^{k,p}(V) \). Since \( T^* \) is bounded, it follows that \( T^*u_n \) is Cauchy in \( W^{k,p}(U) \) and hence convergent. Finally, using the change of variables theorem again we know,

\[
\|T^*u - T^*u_n\|_{L^p(V)}^p \leq \epsilon^{-1} \|u - u_n\|_{L^p(U)}^p \to 0 \text{ as } n \to \infty
\]

and therefore \( T^*u = \lim_{n \to \infty} T^*u_n \) and by continuity Eq. (23.10) still holds for \( u \in W^{k,p}(V) \).

Let \( M \) be a compact \( C^k \) – manifolds without boundary, i.e. \( M \) is a compact Hausdorff space with a collection of charts \( x \) in an “atlas” \( \mathcal{A} \) such that \( x : D(x) \subset \mathbb{R}^d \to M \rightarrow R(x) \subset \mathbb{R}^d \) is a homeomorphism such that

\[
x \circ y^{-1} \in C^k (y (D(x) \cap D(y))), x (D(x) \cap D(y)) \text{ for all } x, y \in \mathcal{A}.
\]

**Definition 23.17.** Let \( \{x_i\}_{i=1}^N \subset \mathcal{A} \) such that \( M = \bigcup_{i=1}^N D(x_i) \) and let \( \{\phi_i\}_{i=1}^N \) be a partition of unity subordinate to the cover \( \{D(x_i)\}_{i=1}^N \). We now define \( u \in W^{k,p}(M) \) if \( u : M \to \mathbb{C} \) is a function such that

\[
(23.11) \quad \|u\|_{W^{k,p}(M)} := \sum_{i=1}^N \| (\phi_i u) \circ x_i^{-1} \|_{W^{k,p}(R(x_i))} < \infty.
\]

Since \( \|\cdot\|_{W^{k,p}(R(x_i))} \) is a norm for all \( i \), it easily verified that \( \|\cdot\|_{W^{k,p}(M)} \) is a norm on \( W^{k,p}(M) \).

**Proposition 23.18.** If \( f \in C^k(M) \) and \( u \in W^{k,p}(M) \) then \( fu \in W^{k,p}(M) \) and

\[
(23.12) \quad \|fu\|_{W^{k,p}(M)} \leq C \|u\|_{W^{k,p}(M)}
\]

where \( C \) is a finite constant not depending on \( u \). Recall that \( f : M \to \mathbb{R} \) is said to be \( C^j \) with \( j \leq k \) if \( f \circ x^{-1} \in C^j(R(x),\mathbb{R}) \) for all \( x \in \mathcal{A} \).

**Proof.** Since \( [f \circ x_i^{-1}] \) has bounded derivatives on \( \text{supp}(\phi_i \circ x_i^{-1}) \), it follows from Proposition 23.6 that there is a constant \( C_i < \infty \) such that

\[
\| (\phi_i f u) \circ x_i^{-1} \|_{W^{k,p}(R(x_i))} = \| [f \circ x_i^{-1}] (\phi_i u) \circ x_i^{-1} \|_{W^{k,p}(R(x_i))} \leq C_i \| (\phi_i u) \circ x_i^{-1} \|_{W^{k,p}(R(x_i))}
\]

and summing this equation on \( i \) shows Eq. (23.12) holds with \( C := \max_i C_i \).

**Theorem 23.19.** If \( \{y_j\}_{j=1}^K \subset \mathcal{A} \) such that \( M = \bigcup_{j=1}^K D(y_j) \) and \( \{\psi_j\}_{j=1}^K \) is a partition of unity subordinate to the cover \( \{D(y_j)\}_{j=1}^K \), then the norm

\[
(23.13) \quad \|u\|_{W^{k,p}(M)} := \sum_{j=1}^K \| (\psi_j u) \circ y_j^{-1} \|_{W^{k,p}(R(y_j))}
\]

is equivalent to the norm in Eq. (23.11). That is to say the space \( W^{k,p}(M) \) along with its topology is well defined independent of the choice of charts and partitions of unity used in defining the norm on \( W^{k,p}(M) \).
Proof.} Since $|u|_{W^{k,p}(M)}$ is a norm,

$$
|u|_{W^{k,p}(M)} = \left| \sum_{i=1}^{N} \phi_i u \right|_{W^{k,p}(M)} \leq \sum_{i=1}^{N} \left| \phi_i u \right|_{W^{k,p}(M)}
$$

$$
= \sum_{j=1}^{K} \left( \sum_{i=1}^{N} \left( \psi_j \phi_i u \right) \cdot y_j^{-1} \right)_{W^{k,p}(R(y_j))}
$$

$$
\leq \sum_{j=1}^{K} \sum_{i=1}^{N} \left( \psi_j \phi_i u \right) \cdot y_j^{-1} \right)_{W^{k,p}(R(y_j))}
$$

(23.14)

and since $x_i \circ y_j^{-1}$ and $y_j \circ x_i^{-1}$ are $C^k$ diffeomorphism and the sets $y_j (\text{supp}(\phi_i) \cap \text{supp}(\psi_j))$ and $x_i (\text{supp}(\phi_i) \cap \text{supp}(\psi_j))$ are compact, an application of Theorem 23.16 and Proposition 23.6 shows there are finite constants $C_{ij}$ such that

$$
\| (\psi_j \phi_i u) \circ y_j^{-1} \|_{W^{k,p}(R(y_j))} \leq C_{ij} \| (\psi_j \phi_i u) \circ x_i^{-1} \|_{W^{k,p}(R(x_i))} \leq C_{ij} \| \phi_i u \circ x_i^{-1} \|_{W^{k,p}(R(x_i))}
$$

which combined with Eq. (23.14) implies

$$
|u|_{W^{k,p}(M)} \leq \sum_{j=1}^{K} \sum_{i=1}^{N} C_{ij} \| \phi_i u \circ x_i^{-1} \|_{W^{k,p}(R(x_i))} \leq C \| u \|_{W^{k,p}(M)}
$$

where $C := \max_{j} \sum_{i=1}^{K} C_{ij} < \infty$. Analogously, one shows there is a constant $K < \infty$ such that $\| u \|_{W^{k,p}(M)} \leq K \| u \|_{W^{k,p}(M)}$. \hfill \blacksquare

Lemma 23.20. Suppose $x \in A(M)$ and $U \subset M$ such that $U \subset \bar{U} \subset D(x)$, then there is a constant $C < \infty$ such that

(23.15) \hspace{1cm} \| u \circ x^{-1} \|_{W^{k,p}(\bar{U})} \leq C \| u \|_{W^{k,p}(M)} \text{ for all } u \in W^{k,p}(M).

Conversely a function $u : M \rightarrow C$ with $\text{supp}(u) \subset U$ is in $W^{k,p}(M)$ iff $\| u \circ x^{-1} \|_{W^{k,p}(\bar{U})} < \infty$ and in any case there is a finite constant such that

(23.16) \hspace{1cm} \| u \|_{W^{k,p}(M)} \leq C \| u \circ x^{-1} \|_{W^{k,p}(\bar{U})}.

Proof.} Choose charts $y_1 := x$, $y_2, \ldots, y_K \in A$ such that $\{ D(y_j) \}_{j=1}^{K}$ is an open cover of $M$ and choose a partition of unity $\{ \psi_j \}_{j=1}^{K}$ subordinate to the cover $\{ D(y_j) \}_{j=1}^{K}$ such that $\psi_1 = 1$ on a neighborhood of $\bar{U}$. To construct such a partition of unity choose $U_j \subset M$ such that $U_j \subset \bar{U} \subset D(y_j)$, $\bar{U} \subset U_1$ and $\bigcup_{j=1}^{K} U_j = M$ and for each $j$ let $\eta_j \in C_c^\infty(D(y_j), [0,1])$ such that $\eta_j = 1$ on a neighborhood of $\bar{U}_j$. Then define $\psi_j := \eta_j (1 - \eta_{j-1}) \cdots (1 - \eta_1)$ where by convention $\eta_0 \equiv 0$. Then $\{ \psi_j \}_{j=1}^{K}$ is the desired partition, indeed by induction one shows

$$
1 - \sum_{j=1}^{l} \psi_j = (1 - \eta_1) \cdots (1 - \eta_l)
$$

and in particular

$$
1 - \sum_{j=1}^{K} \psi_j = (1 - \eta_1) \cdots (1 - \eta_K) = 0.
$$
Using Theorem 23.19, it follows that
\[ \|u \circ x^{-1}\|_{W^{k,p}(U)} = \|(\psi_1 u) \circ x^{-1}\|_{W^{k,p}(U)} \]
\[ \leq \|\psi_j u \circ x^{-1}\|_{W^{k,p}(R(y_j))} \leq \sum_{j=1}^{K} \|\psi_j u \circ y_j^{-1}\|_{W^{k,p}(R(y_j))} \]
\[ = |u|W^{k,p}(M) \leq C \|u\|_{W^{k,p}(M)} \]
which proves Eq. (23.15).

Using Theorems 23.19 and 23.16 there are constants \(C_j\) for \(j = 0, 1, 2, \ldots, N\) such that
\[ \|u\|_{W^{k,p}(M)} \leq C_0 \sum_{j=1}^{K} \|\psi_j u \circ y_j^{-1}\|_{W^{k,p}(R(y_j))} = C_0 \sum_{j=1}^{K} \|\psi_j u \circ y_j^{-1} \circ y_1 \circ y_j^{-1}\|_{W^{k,p}(R(y_j))} \]
\[ \leq C_0 \sum_{j=1}^{K} C_j \|\psi_j u \circ x^{-1}\|_{W^{k,p}(R(y_j))} = C_0 \sum_{j=1}^{K} C_j \|\psi_j \circ x^{-1} \cdot u \circ x^{-1}\|_{W^{k,p}(R(y_j))} \]
This inequality along with \(K\) applications of Proposition 23.6 proves Eq. (23.16).

**Theorem 23.21.** The space \((W^{k,p}(M), \|\cdot\|_{W^{k,p}(M)})\) is a Banach space.

**Proof.** Let \(\{x_i\}_{i=1}^{N} \subset A\) and \(\{\phi_i\}_{i=1}^{N}\) be as in Definition 23.17 and choose \(U_i \subset M\) such that \(\text{supp}(\phi_i) \subset U_i \subset U_i \subset D(x_i)\). If \(\{u_n\}_{n=1}^{\infty} \subset W^{k,p}(M)\) is a Cauchy sequence, then by Lemma 23.20, \(\{u_n \circ x_i^{-1}\}_{n=1}^{\infty} \subset W^{k,p}(x_i(U_i))\) is a Cauchy sequence for all \(i\). Since \(W^{k,p}(x_i(U_i))\) is complete, there exists \(v_i \in W^{k,p}(x_i(U_i))\) such that \(u_n \circ x_i^{-1} \to v_i\) in \(W^{k,p}(x_i(U_i))\). For each \(i\) let \(v_i := \phi_i (v_i \circ x_i)\) and notice by Lemma 23.20 that
\[ \|v_i\|_{W^{k,p}(M)} \leq C \|v_i \circ x_i^{-1}\|_{W^{k,p}(x_i(U_i))} = C \|v_i\|_{W^{k,p}(x_i(U_i))} < \infty \]
so that \(u := \sum_{i=1}^{N} v_i \in W^{k,p}(M)\). Since \(\text{supp}(v_i - \phi_i u_n) \subset U_i\), it follows that
\[ \|u - u_n\|_{W^{k,p}(M)} = \left\| \sum_{i=1}^{N} v_i - \sum_{i=1}^{N} \phi_i u_n \right\|_{W^{k,p}(M)} \]
\[ \leq \sum_{i=1}^{N} \|v_i - \phi_i u_n\|_{W^{k,p}(M)} \leq C \sum_{i=1}^{N} \|\phi_i (v_i \circ x_i - u_n) \circ x_i^{-1}\|_{W^{k,p}(x_i(U_i))} \]
\[ = C \sum_{i=1}^{N} \|\phi_i \circ x_i^{-1} (v_i - u_n \circ x_i^{-1})\|_{W^{k,p}(x_i(U_i))} \]
\[ \leq C \sum_{i=1}^{N} C_i \|\tilde{v}_i - u_n \circ x_i^{-1}\|_{W^{k,p}(x_i(U_i))} \to 0 \text{ as } n \to \infty \]
wherein the last inequality we have used Proposition 23.6 again.

**23.4. Trace Theorems.** For many more general results on this subject matter, see E. Stein [7, Chapter VI].
Lemma 23.22. Suppose $k \geq 1$, $\mathbb{H}^d := \{ x \in \mathbb{R}^d : x_d > 0 \} \subset \text{co} \mathbb{R}^d$, $u \in C^k_c(\mathbb{H}^d)$ and $D$ is the smallest constant so that $\text{supp}(u) \subset \mathbb{R}^{d-1} \times [0, D]$. Then there is a constant $C = C(p, k, D, d)$ such that

$$\| u \|_{W^{k-1,p}(\partial \mathbb{H}^d)} \leq C(p, D, k, d) \| u \|_{W^{k,p}(\mathbb{H}^d)}.$$  

Proof. Write $Q \in \mathbb{H}^d$ as $x = (y, z) \in \mathbb{R}^{d-1} \times [0, \infty)$, then by the fundamental theorem of calculus we have for any $\alpha \in \mathbb{N}^{d-1}_0$ with $|\alpha| \leq k - 1$ that

$$\partial^{\alpha} u(y, 0) = \partial^{\alpha} u(y, z) - \int_0^z \partial^{\alpha} u_t(y, t) \, dt.$$  

Therefore, for $p \in [1, \infty)$

$$|\partial^{\alpha} u(y, 0)|^p \leq 2^{p/q} \cdot \left[ |\partial^{\alpha} u(y, z)|^p + \int_0^z |\partial^{\alpha} u_t(y, t)|^p \, dt \right]^p \leq 2^{p/q} \cdot \left[ |\partial^{\alpha} u(y, z)|^p + \int_0^z |\partial^{\alpha} u_t(y, t)|^p \, dt \cdot |z|^{q/p} \right] \leq 2^{p-1} \cdot \left[ |\partial^{\alpha} u(y, z)|^p + \int_0^D |\partial^{\alpha} u_t(y, t)|^p \, dt \cdot z^{p-1} \right]$$

where $q := \frac{p}{p-1}$ is the conjugate exponent to $p$. Integrating this inequality over $\mathbb{R}^{d-1} \times [0, D]$ implies

$$D \| \partial^{\alpha} u \|_{L^p(\partial \mathbb{H}^d)}^p \leq 2^{p-1} \left[ \| \partial^{\alpha} u \|_{L^p(\mathbb{H}^d)}^p + \| \partial^{\alpha+e_d} u \|_{L^p(\mathbb{H}^d)}^p \frac{D^p}{p} \right]$$

or equivalently that

$$\| \partial^{\alpha} u \|_{L^p(\partial \mathbb{H}^d)}^p \leq 2^{p-1} D^{-1} \| \partial^{\alpha} u \|_{L^p(\mathbb{H}^d)}^p + 2^{p-1} \frac{D^p}{p} \| \partial^{\alpha+e_d} u \|_{L^p(\mathbb{H}^d)}^p$$

from which implies Eq. (23.17).

Similarly, if $p = \infty$, then from Eq. (23.18) we find

$$\| \partial^{\alpha} u \|_{L^\infty(\partial \mathbb{H}^d)} = \| \partial^{\alpha} u \|_{L^\infty(\mathbb{H}^d)} + D \| \partial^{\alpha+e_d} u \|_{L^\infty(\mathbb{H}^d)}$$

and again the result follows. \( \blacksquare \)

Theorem 23.23 (Trace Theorem). Suppose $k \geq 1$ and $\Omega \subset \mathbb{R}^d$ such that $\Omega$ is a compact manifold with $C^k$ – boundary. Then there exists a unique linear map $T : W^{k,p}(\Omega) \rightarrow W^{k-1,p}(\partial \Omega)$ such that $Tu = u|_{\partial \Omega}$ for all $u \in C^k(\Omega)$. \( \circ \)

Proof. Choose a covering $\{ V_i \}_{i=0}^N$ of $\bar{\Omega}$ such that $\bar{V}_0 \subset \Omega$ and for each $i \geq 1$, there is $C^k$ – diffeomorphism $x_i : V_i \rightarrow R(x_i) \subset \text{co} \mathbb{R}^d$ such that

$$x_i(\partial \Omega \cap V_i) = R(x_i) \cap \text{bd}(\mathbb{H}^d)$$

and

$$x_i(\Omega \cap V_i) = R(x_i) \cap \mathbb{H}^d$$

as in Figure 45. Further choose $\phi_i \in C^\infty_c(V_i, [0, 1])$ such that $\sum_{i=0}^N \phi_i = 1$ on a
neighborhood of \( \bar{\Omega} \) and set \( y_i := x_i |_{\partial \Omega \cap V_i} \) for \( i \geq 1 \). Given \( u \in C^k(\bar{\Omega}) \), we compute

\[
\|u|_{\partial \Omega}\|_{W^{k-1,p}(\partial \Omega)} = \sum_{i=1}^{N} \left\| (\phi_i u) |_{\partial \Omega} \circ y_i^{-1} \right\|_{W^{k-1,p}(R(x_i) \cap \text{bd}(\mathbb{H}^d))}
\]
\[
= \sum_{i=1}^{N} \left\| [(\phi_i u) \circ x_i^{-1}] |_{\text{bd}(\mathbb{H}^d)} \right\|_{W^{k-1,p}(R(x_i) \cap \text{bd}(\mathbb{H}^d))}
\]
\[
\leq \sum_{i=1}^{N} C_i \left\| [(\phi_i u) \circ x_i^{-1}] \right\|_{W^{k,p}(R(x_i))}
\]
\[
\leq \max C_i \cdot \sum_{i=1}^{N} \left\| [(\phi_i u) \circ x_i^{-1}] \right\|_{W^{k,p}(R(x_i) \cap \text{bd}(\mathbb{H}^d))} + \left\| [(\phi_0 u) \circ x_0^{-1}] \right\|_{W^{k,p}(R(x_0))}
\]
\[
\leq C \|u\|_{W^{k,p}(\Omega)}
\]

where \( C = \max \{1, C_1, \ldots, C_N\} \). The result now follows by the B.L.T. Theorem 4.1 and the fact that \( C^k(\bar{\Omega}) \) is dense inside \( W^{k,p}(\Omega) \).

**Notation 23.24.** In the sequel will often abuse notation and simply write \( u|_{\partial \Omega} \) for the “function” \( Tu \in W^{k-1,p}(\partial \Omega) \).

**Proposition 23.25** (Integration by parts). Suppose \( \Omega \subset \mathbb{R}^d \) such that \( \bar{\Omega} \) is a compact manifold with \( C^1 \) boundary, \( p \in [1, \infty] \) and \( q = \frac{p}{p-1} \) the conjugate exponent. Then for \( u \in W^{k,p}(\Omega) \) and \( v \in W^{k,q}(\Omega) \),

\[
\int_{\Omega} \partial_i u \cdot v dm = - \int_{\Omega} u \cdot \partial_i v dm + \int_{\partial \Omega} u|_{\partial \Omega} \cdot v|_{\partial \Omega} n_i d\sigma
\]

where \( n : \partial \Omega \rightarrow \mathbb{R}^d \) is unit outward pointing norm to \( \partial \Omega \).
Proof. Equation 23.19 holds for \( u, v \in C^2(\Omega) \) and therefore for \( (u, v) \in W^{k,p}(\Omega) \times W^{k,q}(\Omega) \) since both sides of the equality are continuous in \( (u, v) \in W^{k,p}(\Omega) \times W^{k,q}(\Omega) \) as the reader should verify. 

**Definition 23.26.** Let \( W_0^{k,p}(\Omega) := C_c^\infty(\Omega)^{W^{k,p}(\Omega)} \) be the closure of \( C_c^\infty(\Omega) \) inside \( W^{k,p}(\Omega) \).

**Remark 23.27.** Notice that if \( T : W^{k,p}(\Omega) \to W^{k-1,p}(\partial\Omega) \) is the trace operator in Theorem 23.23, then \( T\left(W_0^{k,p}(\Omega)\right) = \{0\} \subset W^{k-1,p}(\partial\Omega) \) since \( Tu = u|_{\partial\Omega} = 0 \) for all \( u \in C_c^\infty(\Omega) \).

**Corollary 23.28.** Suppose \( \Omega \subset \mathbb{R}^d \) such that \( \bar{\Omega} \) is a compact manifold with \( C^1 \) – boundary, \( p \in [1, \infty] \) and \( T : W^{1,p}(\Omega) \to L^p(\partial\Omega) \) is the trace operator of Theorem 23.23. Then \( W_0^{1,p}(\Omega) = \text{Nul}(T) \).

**Proof.** It has already been observed in Remark 23.27 that \( W_0^{1,p}(\Omega) \subset \text{Nul}(T) \). Suppose \( u \in \text{Nul}(T) \) and \( \text{supp}(u) \) is compactly contained in \( \Omega \). The mollification \( u_\epsilon(x) \) defined in Proposition 23.4 will be in \( C_c^\infty(\Omega) \) for \( \epsilon > 0 \) sufficiently small and by Proposition 23.4, \( u_\epsilon \to u \) in \( W^{1,p}(\Omega) \). Thus \( u \in W_0^{1,p}(\Omega) \). We will now give two proofs for \( \text{Nul}(T) \subset W_0^{1,p}(\Omega) \).

**Proof 1.** For \( u \in \text{Nul}(T) \subset W^{1,p}(\Omega) \) define

\[
\tilde{u}(x) = \begin{cases} 
 u(x) & \text{for } x \in \bar{\Omega} \\
 0 & \text{for } x \notin \bar{\Omega}.
\end{cases}
\]

Then clearly \( \tilde{u} \in L^p(\mathbb{R}^d) \) and moreover by Proposition 23.25, for \( v \in C_c^\infty(\Omega) \),

\[
\int_{\mathbb{R}^d} \tilde{u} \cdot \partial_i vdm = \int_{\Omega} u \cdot \partial_i vdm = -\int_{\Omega} \partial_i u \cdot vdm
\]

from which it follows that \( \partial_i \tilde{u} \) exists weakly in \( L^p(\mathbb{R}^d) \) and \( \partial_i \tilde{u} = 1_{\Omega} \partial_i u \) a.e. Thus \( \tilde{u} \in W^{1,p}(\mathbb{R}^d) \) with \( \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^d)} = \|u\|_{W^{1,p}(\Omega)} \) and \( \text{supp}(\tilde{u}) \subset \Omega \).

Choose \( V \in C_c^1(\mathbb{R}^d,\mathbb{R}^d) \) such that \( V(x) \cdot n(x) > 0 \) for all \( x \in \partial\Omega \) and define

\[
\tilde{u}_\epsilon(x) = T_\epsilon \tilde{u}(x) := \tilde{u} \circ e^{V}(x).
\]

Notice that \( \text{supp}(\tilde{u}_\epsilon) \subset e^{-cV}(\bar{\Omega}) \subset \Omega \) for all \( \epsilon \) sufficiently small. By the change of variables Theorem 23.16, we know that \( \tilde{u}_\epsilon \in W^{1,p}(\Omega) \) and since \( \text{supp}(\tilde{u}_\epsilon) \) is a compact subset of \( \Omega \), it follows from the first paragraph that \( \tilde{u}_\epsilon \in W_0^{1,p}(\Omega) \).

To so finish this proof, it only remains to show \( \tilde{u}_\epsilon \to u \) in \( W^{1,p}(\Omega) \) as \( \epsilon \downarrow 0 \). Looking at the proof of Theorem 23.16, the reader may show there are constants \( \delta > 0 \) and \( C < \infty \) such that

\[
\|T_\epsilon v\|_{W^{1,p}(\mathbb{R}^d)} \leq C \|v\|_{W^{1,p}(\mathbb{R}^d)} \text{ for all } v \in W^{1,p}(\mathbb{R}^d).
\]

By direct computation along with the dominated convergence it may be shown that

\[
T_\epsilon v \to v \text{ in } W^{1,p}(\mathbb{R}^d) \text{ for all } v \in C_c^\infty(\mathbb{R}^d).
\]

As is now standard, Eqs. (23.20) and (23.21) along with the density of \( C_c^\infty(\mathbb{R}^d) \) in \( W^{1,p}(\mathbb{R}^d) \) allows us to conclude \( T_\epsilon v \to v \) in \( W^{1,p}(\mathbb{R}^d) \) for all \( v \in W^{1,p}(\mathbb{R}^d) \) which completes the proof that \( \tilde{u}_\epsilon \to u \) in \( W^{1,p}(\Omega) \) as \( \epsilon \to 0 \).

**Proof 2.** As in the first proof it suffices to show that any \( u \in W_0^{1,p}(\Omega) \) may be approximated by \( v \in W^{1,p}(\Omega) \) with \( \text{supp}(v) \subset \Omega \). As above extend \( u \) to \( \Omega^c \)
by 0 so that \( \tilde{u} \in W^{1,p}(\mathbb{R}^d) \). Using the notation in the proof of 23.23, it suffices to show \( u_i := \tilde{\phi}_i \tilde{u} \in W^{1,p}(\mathbb{R}^d) \) may be approximated by \( u_i \in W^{1,p}(\Omega) \) with \( \text{supp}(u_i) \subset \Omega \). Using the change of variables Theorem 23.16, the problem may be reduced to working with \( w_i = u_i \circ x_i^{-1} \) on \( B = R(x_i) \). But in this case we need only define \( w_i^\epsilon(y) := w_i^\epsilon(y - \epsilon e_d) \) for \( \epsilon > 0 \) sufficiently small. Then \( \text{supp}(w_i^\epsilon) \subset \mathbb{H}^d \cap B \) and as we have already seen \( w_i^\epsilon \to w_i \) in \( W^{1,p}(\mathbb{H}^d) \). Thus \( u_i^\epsilon := w_i^\epsilon \circ x_i \in W^{1,p}(\Omega) \), \( u_i^\epsilon \to u_i \) as \( \epsilon \to 0 \) with \( \text{supp}(u_i) \subset \Omega \). □

23.5. Extension Theorems.

Lemma 23.29. Let \( R > 0 \), \( B := B(0,R) \subset \mathbb{R}^d \), \( B^\pm := \{ x \in B : \pm x_d > 0 \} \) and \( \Gamma := \{ x \in B : x_d = 0 \} \). Suppose that \( u \in C^k(B \setminus \Gamma) \cap C(B) \) and for each \( |\alpha| \leq k \), \( \partial^\alpha u \) extends to a continuous function \( v_\alpha \) on \( B \). Then \( u \in C^k(B) \) and \( \partial^\alpha u = v_\alpha \) for all \( |\alpha| \leq k \).

Proof. For \( x \in \Gamma \) and \( i < d \), then by continuity, the fundamental theorem of calculus and the dominated convergence theorem,

\[
\begin{align*}
u(x + \Delta e_i) - u(x) &= \lim_{y \to x} \int_0^\Delta \partial_i u(y + se_i)ds \\
\end{align*}
\]

and similarly, for \( i = d \),

\[
\begin{align*}
u(x + \Delta e_d) - u(x) &= \lim_{y \to x} \int_0^\Delta \partial_d u(y + se_d)ds \\
\end{align*}
\]

These two equations show, for each \( i \), \( \partial_i u(x) \) exists and \( \partial_i u(x) = v_{\alpha e_i}(x) \). Hence we have shown \( u \in C^1(B) \).

Suppose it has been proven for some \( l \geq 1 \) that \( \partial^\alpha u(x) \) exists and is given by \( v_{\alpha e_l}(x) \) for all \( |\alpha| \leq l < k \). Then applying the results of the previous paragraph to \( \partial^\alpha u(x) \) with \( |\alpha| = l \) shows that \( \partial_l \partial^\alpha u(x) \) exists and is given by \( v_{\alpha + e_l}(x) \) for all \( l \) and \( x \in B \) and from this we conclude that \( \partial^\alpha u(x) \) exists and is given by \( v_{\alpha}(x) \) for all \( |\alpha| \leq l + 1 \). So by induction we conclude \( \partial^\alpha u(x) \) exists and is given by \( v_{\alpha}(x) \) for all \( |\alpha| \leq k \), i.e. \( u \in C^k(B) \). □

Lemma 23.30. Given any \( k + 1 \) distinct points, \( \{ c_i \}_{i=0}^k \), in \( \mathbb{R} \setminus \{ 0 \} \), the \( (k + 1) \times (k + 1) \) matrix \( C \) with entries \( C_{ij} := (c_i)^j \) is invertible.

Proof. Let \( a \in \mathbb{R}^{k+1} \) and define \( p(x) := \sum_{j=0}^k a_j x^j \). If \( a \in \text{Nul}(C) \), then

\[
0 = \sum_{j=0}^k (c_i)^j a_j = p(c_i) \text{ for } i = 0, 1, \ldots, k.
\]

Since \( \deg(p) \leq k \) and the above equation says that \( p \) has \( k + 1 \) distinct roots, we conclude that \( a \in \text{Nul}(C) \) implies \( p \equiv 0 \) which implies \( a = 0 \). Therefore \( \text{Nul}(C) = \{ 0 \} \) and \( C \) is invertible. □
Lemma 23.31. Let $B$, $B^\pm$ and $\Gamma$ be as in Lemma 23.29 and $\{c_i\}_{i=0}^k$, be $k+1$ distinct points in $(\infty, -1]$ for example $c_i = -(i + 1)$ will work. Also let $a \in \mathbb{R}^{k+1}$ be the unique solution (see Lemma 23.30 to $C^{tr}a = 1$ where $1$ denotes the vector of all ones in $\mathbb{R}^{k+1}$, i.e. $a$ satisfies

\[
1 = \sum_{j=0}^k (c_j)^j a_j \text{ for } j = 0, 1, 2 \ldots, k.
\]

For $u \in C^k(\overline{\mathbb{H}^d}) \cap C_c(\overline{\mathbb{H}^d})$ with $\text{supp}(u) \subset B$ and $x = (y, z) \in \mathbb{R}^d$ define

\[
\tilde{u}(x) = \tilde{u}(y, z) = \begin{cases} 
  u(y, z) & \text{if } z \geq 0 \\
  \sum_{i=0}^k a_i u(y, c_i z) & \text{if } z \leq 0.
\end{cases}
\]

Then $\tilde{u} \in C^k(\mathbb{R}^d)$ with $\text{supp}(\tilde{u}) \subset B$ and moreover there exists a constant $M$ independent of $u$ such that

\[
||\tilde{u}||_{W^{k,p}(B)} \leq M ||u||_{W^{k,p}(B^+)}.
\]

Proof. By Eq. (23.22) with $j = 0$,

\[
\sum_{i=0}^k a_i u(y, c_i z) = u(y, 0) \sum_{i=0}^k a_i = u(y, 0).
\]

This shows that $\tilde{u}$ in Eq. (23.23) is well defined and that $\tilde{u} \in C(\overline{\mathbb{H}^d})$. Let $K^- := \{(y, z) : (y, -z) \in \text{supp}(u)\}$. Since $c_i \in (\infty, -1]$, if $x = (y, z) \notin K^-$ and $z < 0$ then $(y, c_i z) \notin \text{supp}(u)$ and therefore $\tilde{u}(x) = 0$ and therefore $\text{supp}(\tilde{u})$ is compactly contained inside of $B$. Similarly if $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, Eq. (23.22) with $j = \alpha_d$ implies

\[
v_\alpha(x) := \begin{cases} 
  (\partial^\alpha u)(y, z) & \text{if } z \geq 0 \\
  \sum_{i=0}^k a_i c_i^\alpha (\partial^\alpha u)(y, c_i z) & \text{if } z \leq 0.
\end{cases}
\]

is well defined and $v_\alpha \in C(\mathbb{R}^d)$. Differentiating Eq. (23.23) shows $\partial^\alpha \tilde{u}(x) = v_\alpha(x)$ for $x \in B \setminus \Gamma$ and therefore we may conclude from Lemma 23.29 that $u \in C^k_c(B) \subset C^k(\mathbb{R}^d)$ and $\partial^\alpha u = v_\alpha$ for all $|\alpha| \leq k$.

We now verify Eq. (23.24) as follows. For $|\alpha| \leq k$,

\[
||\partial^\alpha \tilde{u}||_{L^p(B^-)}^p = \int_{\mathbb{R}^d} 1_{z < 0} \left| \sum_{i=0}^k a_i c_i^\alpha (\partial^\alpha u)(y, c_i z) \right|^p \, dy \, dz \\
\leq C \int_{\mathbb{R}^d} 1_{z < 0} \sum_{i=0}^k |(\partial^\alpha u)(y, c_i z)|^p \, dy \, dz \\
= C \int_{\mathbb{R}^d} 1_{z > 0} \sum_{i=0}^k \frac{1}{|c_i|} |(\partial^\alpha u)(y, z)|^p \, dy \, dz \\
= C \left( \sum_{i=0}^k \frac{1}{|c_i|} \right) ||\partial^\alpha u||_{L^p(B^+)}^p
\]

where $C := \left( \sum_{i=0}^k |a_i c_i^\alpha|^p \right)^{p/q}$ . Summing this equation on $|\alpha| \leq k$ shows there exists a constant $M'$ such that $\|\tilde{u}\|_{W^{k,p}(B^-)} \leq M' \|u\|_{W^{k,p}(B^+)}$ and hence Eq. (23.24) holds with $M = M' + 1$. $\blacksquare$
Theorem 23.32 (Extension Theorem). Suppose \( k \geq 1 \) and \( \Omega \subset \mathbb{R}^d \) such that \( \overline{\Omega} \) is a compact manifold with \( C^k \) boundary. Given \( U \subset \mathbb{R}^d \) such that \( \overline{\Omega} \subset U \), there exists a bounded linear (extension) operator \( E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d) \) such that

(1) \( Eu = u \ a.e. \ in \ \Omega \) and
(2) \( \text{supp}(Eu) \subset U \).

Proof. As in the proof of Theorem 23.23, choose a covering \( \{V_i\}_{i=0}^N \) of \( \overline{\Omega} \) such that \( \overline{V}_0 \subset \Omega \), \( \bigcup_{i=0}^N \overline{V}_i \subset U \) and for each \( i \geq 1 \), there is \( C^k \) diffeomorphism \( x_i : V_i \to R(x_i) \subset \mathbb{R}^d \) such that \( x_i(\partial \Omega \cap V_i) = R(x_i) \cap \text{bd}(\mathbb{H}^d) \) and \( x_i(\Omega \cap V_i) = R(x_i) \cap \mathbb{H}^d = B^+ \)

where \( B^+ \) is as in Lemma 23.31, refer to Figure 45. Further choose \( \phi_i \in C^\infty_c(V_i, [0, 1]) \) such that \( \sum_{i=0}^N \phi_i = 1 \) on a neighborhood of \( \overline{\Omega} \) and set \( y_i := x_i|_{\partial R \setminus V_i} \) for \( i \geq 1 \). Given \( u \in C^k(\overline{\Omega}) \) and \( i \geq 1 \), the function \( v_i := (\phi_i u) \circ x_i^{-1} \) may be viewed as a function in \( C^k(\mathbb{H}^d) \cap C^\infty_c(\mathbb{H}^d) \) with \( \text{supp}(u) \subset B \). Let \( \tilde{v}_i \in C^k_c(B) \) be defined as in Eq. (23.23) above and define \( \tilde{u} := \phi_0 u + \sum_{i=1}^N \tilde{v}_i \circ x_i \in C^\infty_c(\mathbb{R}^d) \) with \( \text{supp}(u) \subset U \). Notice that \( \tilde{u} = u \) on \( \Omega \) and making use of Lemma 23.20 we learn

\[
\|\tilde{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq \|\phi_0 u\|_{W^{k,p}(\mathbb{R}^d)} + \sum_{i=1}^N \|\tilde{v}_i \circ x_i\|_{W^{k,p}(\mathbb{R}^d)} \\
\leq \|\phi_0 u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N \|\tilde{v}_i\|_{W^{k,p}(R(x_i))} \\
\leq C(\phi_0) \|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N C_i \|u\|_{W^{k,p}(B^+)} \\
= C(\phi_0) \|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N \|\phi_i u \circ x_i^{-1}\|_{W^{k,p}(B^+)} \\
\leq C(\phi_0) \|u\|_{W^{k,p}(\Omega)} + \sum_{i=1}^N C_i \|u\|_{W^{k,p}(\Omega)}.
\]

This shows the map \( u \in C^k(\overline{\Omega}) \to Eu := \tilde{u} \in C^k(U) \) is bounded as map from \( W^{k,p}(\Omega) \) to \( W^{k,p}(U) \). As usual, we now extend \( E \) using the B.L.T. Theorem 4.1 to a bounded linear map from \( W^{k,p}(\Omega) \) to \( W^{k,p}(U) \). So for general \( u \in W^{k,p}(\Omega) \), \( Eu = W^{k,p}(U) - \lim_{n \to \infty} \tilde{u}_n \) where \( u_n \in C^k(\overline{\Omega}) \) and \( u = W^{k,p}(\Omega) - \lim_{n \to \infty} u_n \). By passing to a subsequence if necessary, we may assume that \( \tilde{u}_n \) converges a.e. to \( Eu \) from which it follows that \( Eu = u \ a.e. \ on \ \Omega \) and \( \text{supp}(Eu) \subset U \). ■

23.6. Exercises.

Exercise 23.1. Show the norm in Eq. (23.1) is equivalent to the norm

\[
|f|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\Omega)}.
\]
Solution. This is a consequence of the fact that all norms on $l^p \{ \alpha : |\alpha| \leq k \}$ are equivalent. To be more explicit, let $a_\alpha = \| \partial^\alpha f \|_{L^p(\Omega)}$, then

$$
\sum_{|\alpha| \leq k} |a_\alpha| \leq \left( \sum_{|\alpha| \leq k} |a_\alpha|^p \right)^{1/p} \left( \sum_{|\alpha| \leq k} 1^q \right)^{1/q}
$$

while

$$
\left( \sum_{|\alpha| \leq k} |a_\alpha|^p \right)^{1/p} \leq \left( \sum_{|\alpha| \leq k} \left[ \sum_{|\beta| \leq k} |a_\beta| \right]^p \right)^{1/p} \leq \left[ \# \{ \alpha : |\alpha| \leq k \} \right]^{1/p} \sum_{|\beta| \leq k} |a_\beta|.
$$