35. Compact and Fredholm Operators and the Spectral Theorem

In this section \( H \) and \( B \) will be Hilbert spaces. Typically \( H \) and \( B \) will be separable, but we will not assume this until it is needed later.

35.1. Compact Operators.

**Proposition 35.1.** Let \( M \) be a finite dimensional subspace of a Hilbert space \( H \) then

1. \( M \) is complete (hence closed).
2. Closed bounded subsets of \( M \) are compact.

**Proof.** Using the Gram-Schmidt procedure, we may choose an orthonormal basis \( \{\phi_1, \ldots, \phi_n\} \) of \( M \). Define \( U : M \to \mathbb{C}^n \) to be the unique unitary map such that \( U\phi_i = e_i \) where \( e_i \) is the \( i^{th} \) standard basis vector in \( \mathbb{C}^n \). It now follows that \( M \) is complete and that closed bounded subsets of \( M \) are compact since the same is true for \( \mathbb{C}^n \). \( \square \)

**Definition 35.2.** A bounded operator \( K : H \to B \) is **compact** if \( K \) maps bounded sets into precompact sets, i.e. \( K(U) \) is compact in \( B \), where \( U := \{x \in H : \|x\| < 1\} \) is the unit ball in \( H \). Equivalently, for all bounded sequences \( \{x_n\}_{n=1}^\infty \subset H \), the sequence \( \{Kx_n\}_{n=1}^\infty \) has a convergent subsequence in \( B \).

Notice that if \( \dim(H) = \infty \) and \( T : H \to B \) is invertible, then \( T \) is **not** compact.

**Definition 35.3.** \( K : H \to B \) is said to have **finite rank** if \( \text{Ran}(K) \subset B \) is finite dimensional.

**Corollary 35.4.** If \( K : H \to B \) is a finite rank operator, then \( K \) is compact. In particular if either \( \dim(H) < \infty \) or \( \dim(B) < \infty \) then any bounded operator \( K : H \to B \) is finite rank and hence compact.

**Example 35.5.** Let \( (X, \mu) \) be a measure space, \( H = L^2(X, \mu) \) and

\[
k(x, y) \equiv \sum_{i=1}^n f_i(x)g_i(y)
\]

where \( f_i, g_i \in L^2(X, \mu) \) for \( i = 1, \ldots, n \).

Define \( (Kf)(x) = \int_X k(x, y)f(y)d\mu(y) \), then \( K : L^2(X, \mu) \to L^2(X, \mu) \) is a finite rank operator and hence compact.

**Lemma 35.6.** Let \( K := \mathcal{K}(H, B) \) denote the compact operators from \( H \) to \( B \). Then \( \mathcal{K}(H, B) \) is a norm closed subspace of \( L(H, B) \).

**Proof.** The fact that \( \mathcal{K} \) is a vector subspace of \( L(H, B) \) will be left to the reader. Now let \( K_n : H \to B \) be compact operators and \( K : H \to B \) be a bounded operator such that \( \lim_{n \to \infty} \|K_n - K\|_{\text{op}} = 0 \). We will now show \( K \) is compact.

**First Proof.** Given \( \epsilon > 0 \), choose \( N = N(\epsilon) \) such that \( \|K_N - K\| < \epsilon \).

Using the fact that \( K_NU \) is precompact, choose a finite subset \( \Lambda \subset U \) such that \( \min_{x \in \Lambda} \|y - K_Nx\| < \epsilon \) for all \( y \in K_N(U) \). Then for \( z = Kx_0 \in K(U) \) and \( x \in \Lambda \),

\[
\|z - Kx\| = \|(K - K_N)x_0 + K_N(x_0 - x) + (K_N - K)x\|
\leq 2\epsilon + \|K_Nx_0 - K_Nx\|.
\]
Therefore \( \min_{x \in A} \| z - K_N x \| < 3 \epsilon \), which shows \( K(U) \) is 3\( \epsilon \) bounded for all \( \epsilon > 0 \), \( K(U) \) is totally bounded and hence precompact.

**Second Proof.** Suppose \( \{ x_n \}_{n=1}^{\infty} \) is a bounded sequence in \( H \). By compactness, there is a subsequence \( \{ x_{n_k} \}_{k=1}^{\infty} \) of \( \{ x_n \}_{n=1}^{\infty} \) such that \( \{ K x_{n_k} \}_{k=1}^{\infty} \) is convergent in \( B \). Working inductively, we may construct subsequences

\[
\{ x_n \}_{n=1}^{\infty} \supset \{ x_{n_k} \}_{k=1}^{\infty} \supset \{ x_{n_{k_l}} \}_{l=1}^{\infty} \supset \cdots \supset \{ x_{n_{k_m}} \}_{m=1}^{\infty} \supset \cdots
\]

such that \( \{ K x_{n_k} \}_{k=1}^{\infty} \) is convergent in \( B \) for each \( m \). By the usual Cantor’s diagonalization procedure, let \( y_n := x_{n_k} \), then \( \{ y_n \}_{n=1}^{\infty} \) is a subsequence of \( \{ x_n \}_{n=1}^{\infty} \) such that \( \{ K y_n \}_{n=1}^{\infty} \) is convergent for all \( m \). Since

\[
\| K y_n - K y_l \| \leq \| (K - K_m) y_n \| + \| K_m (y_n - y_l) \| + \| (K_m - K) y_l \|
\]

\[
\leq 2 \| K - K_m \| + \| K_m (y_n - y_l) \|,
\]

\[
\lim_{n,l \to \infty} \| K y_n - K y_l \| \leq 2 \| K - K_m \| \to 0 \text{ as } m \to \infty,
\]

which shows \( \{ K y_n \}_{n=1}^{\infty} \) is Cauchy and hence convergent. \( \blacksquare \)

**Proposition 35.7.** A bounded operator \( K : H \to B \) is compact iff there exists finite rank operators, \( K_n : H \to B \), such that \( \| K - K_n \| \to 0 \) as \( n \to \infty \).

**Proof.** Since \( \overline{K(U)} \) is compact it contains a countable dense subset and from this it follows that \( \overline{K(H)} \) is a separable subspace of \( B \). Let \( \{ \phi_n \} \) be an orthonormal basis for \( \overline{K(H)} \subset B \) and \( P_N y = \sum_{n=1}^{N} (y, \phi_n) \phi_n \) be the orthogonal projection of \( y \) onto \( \text{span}\{ \phi_n \}_{n=1}^{N} \). Then \( \lim_{N \to \infty} \| P_N y - y \| = 0 \) for all \( y \in K(H) \).

Define \( K_n = P_n K - \) a finite rank operator on \( H \). For sake of contradiction suppose that \( \limsup_{n \to \infty} \| K - K_n \| = \epsilon > 0 \), in which case there exists \( x_{n_k} \in U \) such that \( \| (K - K_{n_k}) x_{n_k} \| \geq \epsilon \) for all \( n_k \). Since \( K \) is compact, by passing to a subsequence if necessary, we may assume \( \{ K x_{n_k} \}_{n_k=1}^{\infty} \) is convergent in \( B \). Letting \( y \equiv \lim_{k \to \infty} K x_{n_k} \),

\[
\| (K - K_{n_k}) x_{n_k} \| = \| (1 - P_{n_k}) K x_{n_k} \| \leq \| (1 - P_{n_k}) (K x_{n_k} - y) \| + \| (1 - P_{n_k}) y \|
\]

\[
\leq \| K x_{n_k} - y \| + \| (1 - P_{n_k}) y \| \to 0 \text{ as } k \to \infty.
\]

But this contradicts the assumption that \( \epsilon \) is positive and hence we must have \( \lim_{n \to \infty} \| K - K_n \| = 0 \), i.e. \( K \) is an operator norm limit of finite rank operators. The converse direction follows from Corollary 35.4 and Lemma 35.6. \( \blacksquare \)

**Corollary 35.8.** If \( K \) is compact then so is \( K^* \).

**Proof.** Let \( K_n = P_n K \) be as in the proof of Proposition 35.7, then \( K_n^* = K^* P_n \) is still finite rank. Furthermore, using Proposition 12.16,

\[
\| K^* - K_n^* \| = \| K - K_n \| \to 0 \text{ as } n \to \infty
\]

showing \( K^* \) is a limit of finite rank operators and hence compact. \( \blacksquare \)

### 35.2. Hilbert Schmidt Operators.

**Proposition 35.9.** Let \( H \) and \( B \) be a separable Hilbert spaces, \( K : H \to B \) be a bounded linear operator, \( \{ e_n \}_{n=1}^{\infty} \) and \( \{ u_m \}_{m=1}^{\infty} \) be orthonormal basis for \( H \) and \( B \) respectively. Then:
(1) \( \sum_{n=1}^{\infty} \|K e_n\|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2 \) allowing for the possibility that the sums are infinite. In particular the Hilbert Schmidt norm of \( K \),

\[
\|K\|_{HS}^2 := \sum_{n=1}^{\infty} \|K e_n\|^2,
\]

is well defined independent of the choice of orthonormal basis \( \{e_n\}_{n=1}^{\infty} \). We say \( K : H \to B \) is a Hilbert Schmidt operator if \( \|K\|_{HS} < \infty \) and let \( HS(H, B) \) denote the space of Hilbert Schmidt operators from \( H \) to \( B \).

(2) For all \( K \in L(H, B) \), \( \|K\|_{HS} = \|K^*\|_{HS} \) and

\[
\|K\|_{HS} \geq \|K\|_{op} := \sup \{ \|K h\| : h \in H \ni \|h\| = 1 \}.
\]

(3) The set \( HS(H, B) \) is a subspace of \( K(H, B) \) and \( \|\|_{HS} \) is a norm on \( HS(H, B) \) for which \( (HS(H, B), \|\|_{HS}) \) is a Hilbert space. The inner product on \( HS(H, B) \) is given by

\[
(K_1, K_2)_{HS} = \sum_{n=1}^{\infty} (K_1 e_n, K_2 e_n).
\]

(4) Let \( P_N x := \sum_{n=1}^{N} (x, e_n) e_n \) be orthogonal projection onto \( \text{span} \{e_i : i \leq N\} \subset H \) and for \( K \in HS(H, B) \), let \( K_n := KP_n \). Then

\[
\|K - K_N\|^2_{op} \leq \|K - K_N\|^2_{HS} \to 0 \text{ as } N \to \infty,
\]

which shows that finite rank operators are dense in \( (HS(H, B), \|\|_{HS}) \).

(5) If \( L \) is another Hilbert space and \( A : L \to H \) and \( C : B \to L \) are bounded operators, then

\[
\|KA\|_{HS} \leq \|K\|_{HS} \|A\|_{op} \text{ and } \|CK\|_{HS} \leq \|K\|_{HS} \|C\|_{op}.
\]

**Proof.** Items 1. and 2. By Parseval’s equality and Fubini’s theorem for sums,

\[
\sum_{n=1}^{\infty} \|K e_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(Ke_n, u_m)|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |(e_n, K^* u_m)|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2.
\]

This proves \( \|K\|_{HS} \) is well defined independent of basis and that \( \|K\|_{HS} = \|K^*\|_{HS} \). For \( x \in H \setminus \{0\} \), \( x/\|x\| \) may be taken to be the first element in an orthonormal basis for \( H \) and hence

\[
\left\| \frac{K x}{\|x\|} \right\| \leq \|K\|_{HS}.
\]

Multiplying this inequality by \( \|x\| \) shows \( \|K x\| \leq \|K\|_{HS} \|x\| \) and hence \( \|K\|_{op} \leq \|K\|_{HS} \).

Item 3. For \( K_1, K_2 \in L(H, B) \),

\[
\|K_1 + K_2\|_{HS} = \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n + K_2 e_n\|^2}
\]

\[
\leq \sqrt{\sum_{n=1}^{\infty} \left( \|K_1 e_n\|^2 + \|K_2 e_n\|^2 \right)} = \left\{ \|K_1 e_n\| + \|K_2 e_n\| \right\}_{n=1}^{\infty} \|e_n\|_{\ell_2}
\]

\[
\leq \|\{\|K_1 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} + \|\{\|K_2 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} = \|K_1\|_{HS} + \|K_2\|_{HS}.
\]
From this triangle inequality and the homogeneity properties of $\|\cdot\|_{HS}$, we now easily see that $HS(H, B)$ is a subspace of $\mathcal{K}(H, B)$ and $\|\cdot\|_{HS}$ is a norm on $HS(H, B)$. Since

$$
\sum_{n=1}^{\infty} |(K_1 e_n, K_2 e_n)| \leq \sum_{n=1}^{\infty} \|K_1 e_n\| \|K_2 e_n\| \\
\leq \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n\|^2} \sqrt{\sum_{n=1}^{\infty} \|K_2 e_n\|^2} = \|K_1\|_{HS} \|K_2\|_{HS},
$$

the sum in Eq. (35.1) is well defined and is easily checked to define an inner product on $HS(H, B)$ such that $\|K\|_{HS}^2 = (K_1, K_2)_{HS}$. To see that $HS(H, B)$ is complete in this inner product suppose $\{K_m\}_{m=1}^{\infty}$ is a $\|\cdot\|_{HS}$–Cauchy sequence in $HS(H, B)$. Because $L(H, B)$ is complete, there exists $K \in L(H, B)$ such that $\|K_m - K\|_{op} \to 0$ as $m \to \infty$. Since

$$
\sum_{n=1}^{N} \|(K - K_m) e_n\|^2 = \lim_{l \to \infty} \sum_{n=1}^{N} \|(K_l - K_m) e_n\|^2 \leq \lim_{l \to \infty} \|K_l - K_m\|_{HS},
$$

$$
\|K_m - K\|^2_{HS} = \sum_{n=1}^{\infty} \|(K - K_m) e_n\|^2 = \lim_{N \to \infty} \sum_{n=1}^{N} \|(K_l - K_m) e_n\|^2 \\
\leq \lim_{l \to \infty} \|K_l - K_m\|_{HS} \to 0 \text{ as } m \to \infty.
$$

Item 4. Simply observe,

$$
\|K - K_N\|^2_{op} \leq \|K - K_N\|^2_{HS} = \sum_{n>N} \|K e_n\|^2 \to 0 \text{ as } N \to \infty.
$$

Item 5. For $C \in L(B, L)$ and $K \in L(H, B)$ then

$$
\|CK\|^2_{HS} = \sum_{n=1}^{\infty} \|CK e_n\|^2 \leq \|C\|^2_{op} \sum_{n=1}^{\infty} \|K e_n\|^2 = \|C\|^2_{op} \|K\|^2_{HS}
$$

and for $A \in L(L, H)$,

$$
\|KA\|_{HS} = \|A^* K^*\|_{HS} \leq \|A^*\|_{op} \|K^*\|_{HS} = \|A\|_{op} \|K\|_{HS}.
$$

\textbf{Remark 35.10.} The separability assumptions made in Proposition 35.9 are unnecessary. In general, we define

$$
\|K\|^2_{HS} = \sum_{e \in \Gamma} \|K e\|^2
$$

where $\Gamma \subseteq H$ is an orthonormal basis. The same proof of Item 1. of Proposition 35.9 shows $\|K\|_{HS}$ is well defined and $\|K\|_{HS} = \|K^*\|_{HS}$. If $\|K\|^2_{HS} < \infty$, then there exists a countable subset $\Gamma_0 \subseteq \Gamma$ such that $K e = 0$ if $e \in \Gamma \setminus \Gamma_0$. Let $H_0 := \text{span}(\Gamma_0)$ and $B_0 := K(H_0)$. Then $K(H) \subseteq B_0$, $K|_{H_0^\perp} = 0$ and hence by applying the results of Proposition 35.9 to $K|_{H_0} : H_0 \to B_0$ one easily sees that the separability of $H$ and $B$ are unnecessary in Proposition 35.9.
Exercise 35.1. Suppose that $(X, \mu)$ is a $\sigma$–finite measure space such that $H = L^2(X, \mu)$ is separable and $k : X \times X \rightarrow \mathbb{R}$ is a measurable function, such that

$$\|k\|_{L^2(X \times X, \mu \otimes \mu)}^2 = \int_{X \times X} |k(x, y)|^2 d\mu(x) d\mu(y) < \infty.$$ 

Define, for $f \in H$,

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y),$$

when the integral makes sense. Show:

1. $Kf(x)$ is defined for $\mu$–a.e. $x$ in $X$.
2. The resulting function $Kf$ is in $H$ and $K : H \rightarrow H$ is linear.
3. $\|K\|_{HS} = \|k\|_{L^2(X \times X, \mu \otimes \mu)} < \infty$. (This implies $K \in HS(H, H).$)

35.1. Since

$$\int_X d\mu(x) \left( \int_X |k(x, y) f(y)| d\mu(y) \right)^2 \leq \int_X d\mu(x) \left( \int_X |k(x, y)|^2 d\mu(y) \right) \left( \int_X |f(y)|^2 d\mu(y) \right)$$

(35.2)

$$\leq \|k\|_2^2 \|f\|_2^2 < \infty,$$

we learn $Kf$ is almost everywhere defined and that $Kf \in H$. The linearity of $K$ is a consequence of the linearity of the Lebesgue integral. Now suppose $\{\phi_n\}_{n=1}^\infty$ is an orthonormal basis for $H$. From the estimate in Eq. (35.2), $k(x, \cdot) \in H$ for $\mu$–a.e. $x \in X$ and therefore

$$\|K\|_{HS}^2 = \sum_{n=1}^\infty \int_X d\mu(x) \left( \int_X k(x, y) \phi_n(y) d\mu(y) \right)^2$$

$$= \sum_{n=1}^\infty \int_X d\mu(x) \left( \phi_n, k(x, \cdot) \right) = \int_X d\mu(x) \sum_{n=1}^\infty \left( \phi_n, k(x, \cdot) \right)^2$$

$$= \int_X d\mu(x) \|k(x, \cdot)\|_H^2 = \int_X d\mu(x) \int_X d\mu(y) |k(x, y)|^2 = \|k\|_2^2.$$

Example 35.11. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded set, $\alpha < n$, then the operator $K : L^2(\Omega, m) \rightarrow L^2(\Omega, m)$ defined by

$$Kf(x) := \int_\Omega \frac{1}{|x - y|^{\alpha}} f(y) dy$$

is compact.

Proof. For $\epsilon \geq 0$, let

$$K_\epsilon f(x) := \int_\Omega \frac{1}{|x - y|^{\alpha} + \epsilon} f(y) dy = [g_\epsilon * (1_\Omega f)](x)$$

where $g_\epsilon(x) = \frac{1}{|x|^\alpha} 1_C(x)$ with $C \subset \mathbb{R}^n$ a sufficiently large ball such that $\Omega - \Omega \subset C$. Since $\alpha < n$, it follows that

$$g_\epsilon \leq g_0 = |\cdot|^{-\alpha} 1_C \in L^1(\mathbb{R}^n, m).$$

Hence it follows by Proposition 11.12 ?? that

$$\|(K - K_\epsilon) f\|_{L^2(\Omega)} \leq \|(g_0 - g_\epsilon) * (1_\Omega f)\|_{L^2(\mathbb{R}^n)}$$

$$\leq \|(g_0 - g_\epsilon)\|_{L^1(\mathbb{R}^n)} \|1_\Omega f\|_{L^2(\mathbb{R}^n)} = \|(g_0 - g_\epsilon)\|_{L^1(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}.$$
which implies
\[ (35.3) \]
\[ |(K - K_\epsilon)|_{B(L^2(\Omega))} \leq \|g_0 - g_\epsilon\|_{L^1(\mathbb{R}^n)} = \int_{C} \left( \frac{1}{|x|^\alpha + \epsilon} - \frac{1}{|x|^\alpha} \right) dx \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \]
by the dominated convergence theorem. For any \( \epsilon > 0 \),
\[ \int_{\Omega \times \Omega} \left( \frac{1}{|x-y|^\alpha + \epsilon} \right)^2 dxdy < \infty, \]
and hence \( K_\epsilon \) is Hilbert Schmidt and hence compact. By Eq. (35.3), \( K_\epsilon \to K \) as \( \epsilon \downarrow 0 \) and hence it follows that \( K \) is compact as well.

35.3. The Spectral Theorem for Self Adjoint Compact Operators.

**Lemma 35.12.** Suppose \( T : H \to B \) is a bounded operator, then \( \text{Nul}(T^*) = \text{Ran}(T)^\perp \) and \( \text{Ran}(T) = \text{Nul}(T^*)^\perp \).

**Proof.** An element \( y \in B \) is in \( \text{Nul}(T^*) \) iff \( 0 = (T^* y, x) = (y, Ax) \) for all \( x \in H \) which happens iff \( y \in \text{Ran}(T)^\perp \). Because \( \text{Ran}(T) = \text{Ran}(T)^\perp \), \( \text{Ran}(T) = \text{Nul}(T^*)^\perp \).

For the rest of this section, \( T \in \mathcal{K}(H) := \mathcal{K}(H, H) \) will be a self-adjoint compact operator or \textbf{S.A.C.O.} for short.

**Example 35.13 (Model S.A.C.O.).** Let \( H = \ell_2 \) and \( T \) be the diagonal matrix
\[ T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \]
where \( \lim_{n \to \infty} |\lambda_n| = 0 \) and \( \lambda_n \in \mathbb{R} \). Then \( T \) is a self-adjoint compact operator. (Prove!)

The main theorem of this subsection states that up to unitary equivalence, Example 35.13 is essentially the most general example of an S.A.C.O.

**Theorem 35.14.** Suppose \( T \in L(H) := L(H, H) \) is a bounded self-adjoint operator, then
\[ \|T\| = \sup_{f \neq 0} \frac{|(f, Tf)|}{\|f\|^2}. \]
Moreover if there exists a non-zero element \( g \in H \) such that
\[ \frac{|(Tg, g)|}{\|g\|^2} = \|T\|, \]
then \( g \) is an eigenvector of \( T \) with \( Tg = \lambda g \) and \( \lambda \in \{ \pm \|T\| \} \).

**Proof.** Let
\[ M = \sup_{f \neq 0} \frac{|(f, Tf)|}{\|f\|^2}. \]
We wish to show \( M = \|T\| \). Since \( |(f, Tf)| \leq \|f\| \|Tf\| \leq \|T\| \|f\|^2 \), we see \( M \leq \|T\| \).
Conversely let \( f, g \in H \) and compute
\[
(f + g, T(f + g)) - (f - g, T(f - g))
= (f, Tg) + (g, Tf) + (f, Tg) + (g, Tf)
= 2[(f, Tg) + (Tg, f)] = 2[(f, Tg) + (\overline{Tf})]
= 4 \text{Re}(f, Tg).
\]

Therefore, if \( \|f\| = \|g\| = 1 \), it follows that
\[
|\text{Re}(f, Tg)| \leq \frac{M}{4} \{ \|f + g\|^2 + \|f - g\|^2 \} = \frac{M}{4} \{ \|T\|^2 + 2\|g\|^2 \} = M.
\]

By replacing \( f \) be \( e^{i\theta}f \) where \( \theta \) is chosen so that \( e^{i\theta}(f, Tg) \) is real, we find
\[
|\text{Re}(f, Tg)| \leq M \text{ for all } \|f\| = \|g\| = 1.
\]

Hence
\[
\|T\| = \sup_{\|f\| = \|g\| = 1} |(f, Tg)| \leq M.
\]

If \( g \in H \setminus \{0\} \) and \( \|T\| = \|(Tg, g)/\|g\|^2 \) then, using the Cauchy Schwarz inequality,
\[
(35.4) \quad \|T\| = \frac{|(Tg, g)|}{\|g\|^2} \leq \frac{Tg}{\|g\|} \leq \|T\|.
\]

This implies \( \|(Tg, g)\| = \|Tg\|\|g\| \) and forces equality in the Cauchy Schwarz inequality. So by Theorem 12.2, \( Tg \) and \( g \) are linearly dependent, i.e. \( Tg = \lambda g \) for some \( \lambda \in \mathbb{C} \). Substituting this into (35.4) shows that \( |\lambda| = \|T\| \). Since \( T \) is self-adjoint,
\[
\lambda\|g\|^2 = (\lambda g, g) = (Tg, g) = (g, Tg) = (g, \lambda g) = \overline{\lambda}(g, g),
\]
which implies that \( \lambda \in \mathbb{R} \) and therefore, \( \lambda \in \{\pm \|T\|\} \).

**Theorem 35.15.** Let \( T \) be a S.A.C.O., then either \( \lambda = \|T\| \) or \( \lambda = -\|T\| \) is an eigenvalue of \( T \).

**Proof.** Without loss of generality we may assume that \( T \) is non-zero since otherwise the result is trivial. By Theorem 35.14, there exists \( f_n \in H \) such that \( \|f_n\| = 1 \) and
\[
(35.5) \quad \frac{|(f_n, T f_n)|}{\|f_n\|^2} = \frac{|(f_n, T f_n)|}{\|f_n\|^2} \longrightarrow \|T\| \text{ as } n \to \infty.
\]

By passing to a subsequence if necessary, we may assume that \( \lambda := \lim_{n \to \infty} (f_n, T f_n) \) exists and \( \lambda \in \{\pm \|T\|\} \). By passing to a further subsequence if necessary, we may assume, using the compactness of \( T \), that \( T f_n \) is convergent as well. We now compute:
\[
0 \leq \|T f_n - \lambda f_n\|^2 = \|T f_n\|^2 - 2\lambda(T f_n, f_n) + \lambda^2
\leq \lambda^2 - 2\lambda(T f_n, f_n) + \lambda^2 \to \lambda^2 - 2\lambda^2 + \lambda^2 = 0 \text{ as } n \to \infty.
\]

Hence
\[
(35.6) \quad T f_n - \lambda f_n \to 0 \text{ as } n \to \infty
\]
and therefore
\[
f \equiv \lim_{n \to \infty} f_n = \frac{1}{\lambda} \lim_{n \to \infty} T f_n
\]
exists. By the continuity of the inner product, \( ||f|| = 1 \neq 0 \). By passing to the limit in Eq. (35.6) we find that \( Tf = \lambda f \).

**Lemma 35.16.** Let \( T : H \to H \) be a self-adjoint operator and \( M \) be a \( T \)-invariant subspace of \( H \), i.e. \( T(M) \subset M \). Then \( M^\perp \) is also a \( T \)-invariant subspace, i.e. \( T(M^\perp) \subset M^\perp \).

**Proof.** Let \( x \in M \) and \( y \in M^\perp \), then \( Tx \in M \) and hence

\[
0 = (Tx, y) = (x, Ty) \text{ for all } x \in M.
\]

Thus \( Ty \in M^\perp \). ■

**Theorem 35.17** (Spectral Theorem). Suppose that \( T : H \to H \) is a non-zero S.A.C.O., then

1. there exists at least one eigenvalue \( \lambda \in \{ \pm ||T|| \} \).
2. There are at most countable many non-zero eigenvalues, \( \{ \lambda_n \}_{n=1}^N \), where \( N = \infty \) is allowed. (Unless \( T \) is finite rank, \( N \) will be infinite.)
3. The \( \lambda_n \)'s (including multiplicities) may be arranged so that \( |\lambda_n| \geq |\lambda_{n+1}| \) for all \( n \). If \( N = \infty \) then \( \lim_{n \to \infty} |\lambda_n| = 0 \). (In particular any eigenspace for \( T \) with non-zero eigenvalue is finite dimensional.)
4. The eigenvectors \( \{ \phi_n \}_{n=1}^N \) can be chosen to be an O.N. set such that \( H = \text{span}\{\phi_n\} \oplus \text{null}(T) \).
5. Using the \( \{ \phi_n \}_{n=1}^N \) above,

\[
T \psi = \sum_{n=1}^N \lambda_n (\psi, \phi_n) \phi_n \text{ for all } \psi \in H.
\]

6. The spectrum of \( T \) is \( \sigma(T) = \{0\} \cup \cup_{n=1}^\infty \{ \lambda_n \} \).

**Proof.** We will find \( \lambda_n \)'s and \( \phi_n \)'s recursively. Let \( \lambda_1 \in \{ \pm ||T|| \} \) and \( \phi_1 \in H \) such that \( T \phi_1 = \lambda_1 \phi_1 \) as in Theorem 35.15. Take \( M_1 = \text{span}(\phi_1) \) so \( T(M_1) \subset M_1 \).

By Lemma 35.16, \( TM_1^\perp \subset M_1^\perp \). Define \( T_1 : M_1^\perp \to M_1^\perp \) via \( T_1 = T|_{M_1^\perp} \). Then \( T_1 \) is again a compact operator. If \( T_1 = 0 \), we are done.

If \( T_1 \neq 0 \), by Theorem 35.15 there exists \( \lambda_2 \in \{ \pm ||T|| \} \) and \( \phi_2 \in M_1^\perp \) such that \( ||\phi_2|| = 1 \) and \( T_1 \phi_2 = T \phi_2 = \lambda_2 \phi_2 \). Let \( M_2 = \text{span}(\phi_1, \phi_2) \). Again \( T(M_2) \subset M_2 \) and hence \( T_2 = T|_{M_2^\perp} : M_2^\perp \to M_2^\perp \) is compact. Again if \( T_2 = 0 \) we are done.

If \( T_2 \neq 0 \). Then by Theorem 35.15 there exists \( \lambda_3 \in \{ \pm ||T|| \} \) and \( \phi_3 \in M_2^\perp \) such that \( ||\phi_3|| = 1 \) and \( T_2 \phi_3 = T \phi_3 = \lambda_3 \phi_3 \). Continuing this way indefinitely or until we reach a point where \( T_n = 0 \), we construct a sequence \( \{ \lambda_n \}_{n=1}^\infty \) of eigenvalues and orthonormal eigenvectors \( \{ \phi_n \}_{n=1}^\infty \) such that \( |\lambda_i| \geq |\lambda_{i+1}| \) with the further property that

\[
|\lambda_i| = \sup_{\phi \perp \{\phi_1, \phi_2, \ldots, \phi_{i-1}\}} \frac{||T \phi||}{||\phi||}
\]

If \( N = \infty \) then \( \lim_{n \to \infty} |\lambda_n| = 0 \) for if not there would exist \( \epsilon > 0 \) such that \( |\lambda_i| \geq \epsilon > 0 \) for all \( i \). In this case \( \{\phi_i/\lambda_i\}_{i=1}^\infty \) is sequence in \( H \) bounded by \( \epsilon^{-1} \). By compactness of \( T \), there exists a subsequence \( i_k \) such that \( \phi_{i_k} = T \phi_{i_k}/\lambda_{i_k} \) is convergent. But this is impossible since \( \{ \phi_{i_k} \} \) is an orthonormal set. Hence we must have that \( \epsilon = 0 \).
Let $M \equiv \text{span}\{\phi_i\}_{i=1}^{N}$ with $N = \infty$ possible. Then $T(M) \subset M$ and hence $T(M^\perp) \subset M^\perp$. Using Eq. (35.7),

$$
\|T|_{M^\perp}\| \leq \|T|_{M^\perp}\| = |\lambda_n| \longrightarrow 0 \text{ as } n \rightarrow \infty
$$

showing $T|M^\perp = 0$.

Define $P_0$ to be orthogonal projection onto $M^\perp$. Then for $\psi \in H$,

$$
\psi = P_0\psi + (1 - P_0)\psi = P_0\psi + \sum_{i=1}^{N} (\psi, \phi_i)\phi_i
$$

and

$$
T\psi = TP_0\psi + T\sum_{i=1}^{N} (\psi, \phi_i)\phi_i = \sum_{i=1}^{N} \lambda_i(\psi, \phi_i)\phi_i.
$$

Since $\{\lambda_n\} \subset \sigma(T)$ and $\sigma(T)$ is closed, it follows that $0 \in \sigma(T)$ and hence $\{\lambda_n\}_{n=1}^{\infty} \cup \{0\} \subset \sigma(T)$. Suppose that $z \notin \{\lambda_n\}_{n=1}^{\infty} \cup \{0\}$ and let $d$ be the distance between $z$ and $\{\lambda_n\}_{n=1}^{\infty} \cup \{0\}$. Notice that $d > 0$ because $\lim_{n \to \infty} \lambda_n = 0$. A few simple computations show that:

$$
(T - zI)\psi = \sum_{i=1}^{N} (\psi, \phi_i)(\lambda_i - z)\phi_i - zP_0\psi,
$$

$(T - z)^{-1}$ exists,

$$
(T - zI)^{-1}\psi = \sum_{i=1}^{N} (\psi, \phi_i)(\lambda_i - z)^{-1}\phi_i - z^{-1}P_0\psi,
$$

and

$$
\|(T - zI)^{-1}\psi\|^2 = \sum_{i=1}^{N} |(\psi, \phi_i)|^2 \left( \frac{1}{|\lambda_i - z|^2} \right) + \frac{1}{|z|^2} \|P_0\psi\|^2 \\
\leq \left( \frac{1}{d^2} \right)^2 \left( \sum_{i=1}^{N} |(\psi, \phi_i)|^2 + \|P_0\psi\|^2 \right) = \frac{1}{d^2} \|\psi\|^2.
$$

We have thus shown that $(T - zI)^{-1}$ exists, $\|(T - zI)^{-1}\| \leq d^{-1} < \infty$ and hence $z \notin \sigma(T)$.

35.4. Structure of Compact Operators.

**Theorem 35.18.** Let $K : H \to B$ be a compact operator. Then there exists $N \in \mathbb{N} \cup \{\infty\}$, orthonormal subsets $\{\phi_n\}_{n=1}^{N} \subset H$ and $\{\psi_n\}_{n=1}^{N} \subset B$ and a sequences $\{\lambda_n\}_{n=1}^{N} \subset \mathbb{C}$ such that $\lim_{n \to \infty} \lambda_n = 0$ if $N = \infty$ and

$$
Kf = \sum_{n=1}^{N} \lambda_n(f, \phi_n)\psi_n \text{ for all } f \in H.
$$

**Proof.** The operator $K^*K \in K(H)$ is self-adjoint and hence by Theorem 35.17, there exists an orthonormal set $\{\phi_n\}_{n=1}^{N} \subset H$ and $\{\mu_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that

$$
K^*Kf = \sum_{n=1}^{N} \mu_n(f, \phi_n)\phi_n \text{ for all } f \in H.
Let \( \lambda_n := \sqrt{\mu_n} \) and \( \sqrt{K^*K} \in \mathcal{K}(H) \) be defined by
\[
\sqrt{K^*K} f = \sum_{n=1}^{N} \lambda_n (f, \phi_n) \phi_n \quad \text{for all} \quad f \in H.
\]

Define \( U \in L(H, B) \) so that \( U = \"K (K^*K)^{-1/2} \", \) or more precisely by
\[
(35.8) \quad U f = \sum_{n=1}^{N} \lambda_n^{-1} (f, \phi_n) K \phi_n.
\]

The operator \( U \) is well defined because
\[
(\lambda_n^{-1} K \phi_n, \lambda_m^{-1} K \phi_m) = \lambda_n^{-1} \lambda_m^{-1} (\phi_n, K^* K \phi_m) = \lambda_n^{-1} \lambda_m^{-1} \lambda_m \delta_{m,n} = \delta_{m,n}
\]
which shows \( \{\lambda_n^{-1} K \phi_n\}_{n=1}^{\infty} \) is an orthonormal subset of \( B \). Moreover this also shows
\[
\| U f \|_2^2 = \sum_{n=1}^{N} |(f, \phi_n)|^2 = \| P f \|_2^2
\]
where \( P = P_{\text{Null}(K^*)} \). Replacing \( f \) by \( (K^*K)^{1/2} f \) in Eq. (35.8) shows
\[
(35.9) \quad U (K^*K)^{1/2} f = \sum_{n=1}^{N} \lambda_n^{-1} ((K^*K)^{1/2} f, \phi_n) K \phi_n = \sum_{n=1}^{N} (f, \phi_n) K \phi_n = K f,
\]
since \( f = \sum_{n=1}^{N} (f, \phi_n) \phi_n + P f \).

From Eq. (35.9) it follows that
\[
K f = \sum_{n=1}^{N} \lambda_n (f, \phi_n) U \phi_n = \sum_{n=1}^{N} \lambda_n (f, \phi_n) U \phi_n
\]
where \( \{\psi_n\}_{n=1}^{N} \) is the orthonormal sequence in \( B \) defined by
\[
\psi_n := U \phi_n = \lambda_n^{-1} K \phi_n.
\]

35.4.1. Trace Class Operators. We will say \( K \in \mathcal{K}(H) \) is trace class if
\[
\text{tr}(\sqrt{K^*K}) := \sum_{n=1}^{N} \lambda_n < \infty
\]
in which case we define
\[
\text{tr}(K) = \sum_{n=1}^{N} \lambda_n (\psi_n, \phi_n).
\]

Notice that if \( \{e_m\}_{m=1}^{\infty} \) is any orthonormal basis in \( H \) (or for the \( \overline{\text{Ran}(K)} \) if \( H \) is not separable) then
\[
\sum_{m=1}^{M} (Ke_m, e_m) = \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda_n (e_m, \phi_n) \psi_n, e_m) = \sum_{n=1}^{N} \lambda_n \sum_{m=1}^{M} (e_m, \phi_n) (\psi_n, e_m)
\]
\[
= \sum_{n=1}^{N} \lambda_n (P_M \psi_n, \phi_n)
\]
where $P_M$ is orthogonal projection onto $\text{Span}(e_1,\ldots,e_M)$. Therefore by dominated convergence theorem,

$$
\sum_{m=1}^{\infty} (Ke_m, e_m) = \lim_{M \to \infty} \sum_{n=1}^{N} \lambda_n (P_M \psi_n, \phi_n) = \sum_{n=1}^{N} \lambda_n \lim_{M \to \infty} (P_M \psi_n, \phi_n) = \sum_{n=1}^{N} \lambda_n (\psi_n, \phi_n) = \text{tr}(K).
$$

35.5. Fredholm Operators.

**Lemma 35.19.** Let $M \subset H$ be a closed subspace and $V \subset H$ be a finite dimensional subspace. Then $M + V$ is closed as well. In particular if $\text{codim}(M) = \dim(H/M) < \infty$ and $W \subset H$ is a subspace such that $M \subset W$, then $W$ is closed and $\text{codim}(W) < \infty$.

**Proof.** Let $P : H \to M$ be orthogonal projection and let $V_0 := (I - P)V$. Since $\dim(V_0) \leq \dim(V) < \infty$, $V_0$ is still closed. Also it is easily seen that $M + V = M \oplus V_0$

from which it follows that $M + V$ is closed because $\{z_n = m_n + v_n \subset M \oplus V_0$ is convergent if $\{m_n\} \subset M$ and $\{v_n\} \subset V_0$ are convergent.

If $\text{codim}(M) < \infty$ and $M \subset W$, there is a finite dimensional subspace $V \subset H$ such that $W = M + V$ and so by what we have just proved, $W$ is closed as well. It should also be clear that $\text{codim}(W) \leq \text{codim}(M) < \infty$. ■

**Lemma 35.20.** If $K : H \to B$ is a finite rank operator, then there exists $\{\phi_n\}_{n=1}^{k} \subset H$ and $\{\psi_n\}_{n=1}^{k} \subset B$ such that

1. $Kx = \sum_{n=1}^{k} (x, \phi_n) \psi_n$ for all $x \in H$.
2. $K^*y = \sum_{n=1}^{k} (y, \psi_n) \phi_n$ for all $y \in B$, in particular $K^*$ is still finite rank.

For the next two items, further assume $B = H$.

3. $\dim \text{Nul}(I + K) < \infty$.
4. $\dim \text{coker}(I + K) < \infty$, $\text{Ran}(I + K)$ is closed and

$$
\text{Ran}(I + K) = \text{Nul}(I + K)^\perp.
$$

**Proof.**

1. Choose $\{\psi_n\}_{n=1}^{k}$ to be an orthonormal basis for $\text{Ran}(K)$. Then for $x \in H$,

$$
Kx = \sum_{n=1}^{k} (Kx, \psi_n) \psi_n = \sum_{n=1}^{k} (x, K^* \psi_n) \psi_n = \sum_{n=1}^{k} (x, \phi_n) \psi_n
$$

where $\phi_n \equiv K^* \psi_n$.

2. Item 2. is a simple computation left to the reader.

3. Since $\text{Nul}(I + K) = \{x \in H \mid x = -Kx\} \subset \text{Ran}(K)$ it is finite dimensional.

4. Since $x = (I + K)x \in \text{Ran}(I + K)$ for $x \in \text{Nul}(K)$, $\text{Nul}(K) \subset \text{Ran}(I + K)$.

Since $\{\phi_1, \phi_2, \ldots, \phi_k\}^\perp \subset \text{Nul}(K)$, $H = \text{Nul}(K) + \text{span} \{\phi_1, \phi_2, \ldots, \phi_k\}$ and thus $\text{codim} \text{(Nul}(K)) < \infty$. From these comments and Lemma 35.19, $\text{Ran}(I + K)$ is closed and $\text{codim} \text{(Ran}(I + K)) \leq \text{codim} \text{(Nul}(K)) < \infty$. The assertion that $\text{Ran}(I + K) = \text{Nul}(I + K)^\perp$ is a consequence of Lemma 35.12 below.
Definition 35.21. A bounded operator $F : H \to B$ is Fredholm iff the \( \dim \text{Nul}(F) < \infty \), \( \dim \text{coker}(F) < \infty \) and \( \text{Ran}(F) \) is closed in \( B \). (Recall: \( \text{coker}(F) \equiv B/\text{Ran}(F) \). ) The index of \( F \) is the integer,

\begin{align}
\text{index}(F) &= \dim \text{Nul}(F) - \dim \text{coker}(F) \\
\text{dim}(F) &= \dim \text{Nul}(F) - \dim \text{Nul}(F^*)
\end{align}

Notice that equations (35.10) and (35.11) are the same since, (using \( \text{Ran}(F) \) is closed)

\[ B = \text{Ran}(F) \oplus \text{Ran}(F)^\perp = \text{Ran}(F) \oplus \text{Nul}(F^*) \]

so that \( \text{coker}(F) = B/\text{Ran}(F) \cong \text{Nul}(F^*) \).

Lemma 35.22. The requirement that \( \text{Ran}(F) \) is closed in Definition 35.21 is redundant.

Proof. By restricting \( F \) to \( \text{Nul}(F)^\perp \), we may assume without loss of generality that \( \text{Nul}(F) = \{0\} \). Assuming \( \dim \text{coker}(F) < \infty \), there exists a finite dimensional subspace \( V \subset B \) such that \( B = \text{Ran}(F) \oplus V \). Since \( V \) is finite dimensional, \( V \) is closed and hence \( B = V \oplus V^\perp \). Let \( \pi : B \to V^\perp \) be the orthogonal projection operator onto \( V^\perp \) and let \( G \equiv \pi F : H \to V^\perp \) which is continuous, being the composition of two bounded transformations. Since \( G \) is a linear isomorphism, as the reader should check, the open mapping theorem implies the inverse operator \( G^{-1} : V^\perp \to H \) is bounded.

Suppose that \( h_n \in H \) is a sequence such that \( \lim_{n \to \infty} F(h_n) = b \) exists in \( B \). Then by composing this last equation with \( \pi \), we find that \( \lim_{n \to \infty} G(h_n) = \pi(b) \) exists in \( V^\perp \). Composing this equation with \( G^{-1} \) shows that \( h := \lim_{n \to \infty} h_n = G^{-1}\pi(b) \) exists in \( H \). Therefore, \( F(h_n) \to F(h) \in \text{Ran}(F) \), which shows that \( \text{Ran}(F) \) is closed. \qed

Remark 35.23. It is essential that the subspace \( M \equiv \text{Ran}(F) \) in Lemma 35.22 is the image of a bounded operator, for it is not true that every finite codimensional subspace \( M \) of a Banach space \( B \) is necessarily closed. To see this suppose that \( B \) is a separable infinite dimensional Banach space and let \( A \subset B \) be an algebraic basis for \( B \), which exists by a Zorn's lemma argument. Since \( \dim(B) = \infty \) and \( B \) is complete, \( A \) must be uncountable. Indeed, if \( A \) were countable we could write \( B = \bigcup_{n=1}^{\infty} B_n \) where \( B_n \) are finite dimensional (necessarily closed) subspaces of \( B \). This shows that \( B \) is the countable union of nowhere dense closed subsets which violates the Baire Category theorem.

By separability of \( B \), there exists a countable subset \( A_0 \subset A \) such that the closure of \( M_0 \equiv \text{span}(A_0) \) is equal to \( B \). Choose \( x_0 \in A \setminus A_0 \), and let \( M \equiv \text{span}(A \setminus \{x_0\}) \). Then \( M_0 \subset M \) so that \( B = \bar{M}_0 = \bar{M} \), while \( \text{codim}(M) = 1 \). Clearly this \( M \) can not be closed.

Example 35.24. Suppose that \( H \) and \( B \) are finite dimensional Hilbert spaces and \( F : H \to B \) is Fredholm. Then

\begin{equation}
\text{index}(F) = \dim(B) - \dim(H).
\end{equation}

The formula in Eq. (35.12) may be verified using the rank nullity theorem,

\[ \dim(H) = \dim \text{Nul}(F) + \dim \text{Ran}(F), \]

and the fact that

\[ \dim(B/\text{Ran}(F)) = \dim(B) - \dim \text{Ran}(F). \]
Theorem 35.25. A bounded operator $F : H \to B$ is Fredholm iff there exists a bounded operator $A : B \to H$ such that $AF - I$ and $FA - I$ are both compact operators. (In fact we may choose $A$ so that $AF - I$ and $FA - I$ are both finite rank operators.)

Proof. ($\Rightarrow$) Suppose $F$ is Fredholm, then $F : \text{Nul}(F)^\perp \to \text{Ran}(F)$ is a bijective bounded linear map between Hilbert spaces. (Recall that $\text{Ran}(F)$ is a closed subspace of $B$ and hence a Hilbert space.) Let $\bar{F}$ be the inverse of this map—a bounded map by the open mapping theorem. Let $P : H \to \text{Ran}(F)$ be orthogonal projection and set $A \equiv \bar{F}P$. Then $AF - I = \bar{F}PF - I = \bar{F}F - I = -Q$ where $Q$ is the orthogonal projection onto $\text{Nul}(F)$. Similarly, $FA - I = F\bar{F}P - I = -(I - P)$. Because $I - P$ and $Q$ are finite rank projections and hence compact, both $AF - I$ and $FA - I$ are compact.

($\Leftarrow$) We first show that the operator $A : B \to H$ may be modified so that $AF - I$ and $FA - I$ are both finite rank operators. To this end let $G \equiv AF - I$ ($G$ is compact) and choose a finite rank approximation $G_1$ to $G$ such that $G = G_1 + \mathcal{E}$ where $||\mathcal{E}|| < 1$. Define $A_L : B \to H$ to be the operator $A_L \equiv (I + \mathcal{E})^{-1}A$. Since $AF = (I + \mathcal{E})^{-1}AF = I + (I + \mathcal{E})^{-1}G_1 = I + K_L$

where $K_L$ is a finite rank operator. Similarly there exists a bounded operator $A_R : B \to H$ and a finite rank operator $K_R$ such that $FA_R = I + K_R$. Notice that $A_LFA_R = A_R + K_LA_R$ on one hand and $A_LFA_R = A_L + A_LK_R$ on the other. Therefore, $A_L - A_R = A_LK_R - K.LA_R =: S$ is a finite rank operator. Therefore $FA_L - I = K_R - FS$ is still a finite rank operator. Thus we have shown that there exists a bounded operator $A : B \to H$ such that $AF - I$ and $FA - I$ are both finite rank operators.

We now assume that $A$ is chosen such that $AF - I = G_1$, $FA - I = G_2$ are finite rank. Clearly $\text{Nul}(F) \subset \text{Nul}(AF) = \text{Nul}(I + G_1)$ and $\text{Ran}(F) \supset \text{Ran}(FA) = \text{Ran}(I + G_2)$. The theorem now follows from Lemma 35.19 and Lemma 35.20. ■

Corollary 35.26. If $F : H \to B$ is Fredholm then $F^*$ is Fredholm and $\text{index}(F) = -\text{index}(F^*)$.

Proof. Choose $A : B \to H$ such that both $AF - I$ and $FA - I$ are compact. Then $F^*A^* - I$ and $A^*F^* - I$ are compact which implies that $F^*$ is Fredholm. The assertion, $\text{index}(F) = -\text{index}(F^*)$, follows directly from Eq. (35.11). ■

Lemma 35.27. A bounded operator $F : H \to B$ is Fredholm if and only if there exists orthogonal decompositions $H = H_1 \oplus H_2$ and $B = B_1 \oplus B_2$ such that

1. $H_1$ and $B_1$ are closed subspaces,
2. $H_2$ and $B_2$ are finite dimensional subspaces, and
3. $F$ has the block diagonal form

\[
F = \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix} : H_1 \oplus H_2 \rightarrow B_1 \oplus B_2
\]

with $F_{11} : H_1 \to B_1$ being a bounded invertible operator.

Furthermore, given this decomposition, $\text{index}(F) = \dim(H_2) - \dim(B_2)$.
Proof. If $F$ is Fredholm, set $H_1 = \text{Nul}(F)\perp, H_2 = \text{Nul}(F), B_1 = \text{Ran}(F),$ and $B_2 = \text{Ran}(F)\perp$. Then $F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$, where $F_{11} \equiv F|_{H_1} : H_1 \to B_1$ is invertible.

For the converse, assume that $F$ is given as in Eq. (35.13). Let $A \equiv \begin{pmatrix} F_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ then
\[
AF = \begin{pmatrix} I & F_{11}^{-1}F_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & F_{11}^{-1}F_{12} \\ 0 & -I \end{pmatrix},
\]
so that $AF - I$ is finite rank. Similarly one shows that $FA - I$ is finite rank, which shows that $F$ is Fredholm.

Now to compute the index of $F$, notice that \[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \text{Nul}(F) \iff
F_{11}x_1 + F_{12}x_2 = 0
F_{21}x_1 + F_{22}x_2 = 0
\]
which happens iff $x_1 = -F_{11}^{-1}F_{12}x_2$ and $(-F_{21}F_{11}^{-1}F_{12} + F_{22})x_2 = 0$. Let $D \equiv (F_{22} - F_{21}F_{11}^{-1}F_{12}) : H_2 \to B_2$, then the mapping \[
x_2 \in \text{Nul}(D) \to \begin{pmatrix} -F_{11}^{-1}F_{12}x_2 \\ x_2 \end{pmatrix} \in \text{Nul}(F)
\]
is a linear isomorphism of vector spaces so that $\text{Nul}(F) \cong \text{Nul}(D)$. Since
\[
F^* = \begin{pmatrix} F_{11}^{*} & F_{12}^{*} \\ F_{21}^{*} & F_{22}^{*} \end{pmatrix}
\]
B_1 \oplus \to H_1 \oplus, B_2 \rightarrow H_2
\]
similar reasoning implies $\text{Nul}(F^*) \cong \text{Nul}(D^*)$. This shows that $\text{index}(F) = \text{index}(D)$. But we have already seen in Example 35.24 that $\text{index}(D) = \dim H_2 - \dim B_2$. $

\textbf{Proposition 35.28.}$ Let $F$ be a Fredholm operator and $K$ be a compact operator from $H \to B$. Further assume $T : B \to X$ (where $X$ is another Hilbert space) is also Fredholm. Then

(1) the Fredholm operators form an open subset of the bounded operators. Moreover if $\mathcal{E} : H \to B$ is a bounded operator with $\|\mathcal{E}\|$ sufficiently small we have $\text{index}(F) = \text{index}(F + \mathcal{E})$.

(2) $F + K$ is Fredholm and $\text{index}(F) = \text{index}(F + K)$.

(3) $TF$ is Fredholm and $\text{index}(TF) = \text{index}(T) + \text{index}(F)$

\textbf{Proof.}

(1) We know $F$ may be written in the block form given in Eq. (35.13) with $F_{11} : H_1 \to B_1$ being a bounded invertible operator. Decompose $\mathcal{E}$ into the block form as
\[
\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}
\]
and choose $\|\mathcal{E}\|$ sufficiently small such that $\|\mathcal{E}_{11}\|$ is sufficiently small to guarantee that $F_{11} + \mathcal{E}_{11}$ is still invertible. (Recall that the invertible operators form an open set.) Thus $F + \mathcal{E} = \begin{pmatrix} F_{11} + \mathcal{E}_{11} & * \\ * & * \end{pmatrix}$ has the block
form of a Fredholm operator and the index may be computed as:

\[ \text{index}(F + \mathcal{E}) = \dim H_2 - \dim B_2 = \text{index}(F). \]

(2) Given \( K : H \to B \) compact, it is easily seen that \( F + K \) is still Fredholm. Indeed if \( A : B \to H \) is a bounded operator such that \( G_1 \equiv AF - I \) and \( G_2 \equiv FA - I \) are both compact, then \( A(F + K) - I = G_1 + AK \) and \( (F + K)A - I = G_2 + KA \) are both compact. Hence \( F + K \) is Fredholm by Theorem 35.25. By item 1., the function \( f(t) \equiv \text{index}(F + tK) \) is a continuous locally constant function of \( t \in \mathbb{R} \) and hence is constant. In particular, \( \text{index}(F + K) = f(1) = f(0) = \text{index}(F) \).

(3) It is easily seen, using Theorem 35.25 that the product of two Fredholm operators is again Fredholm. So it only remains to verify the index formula in item 3.

For this let \( H_1 \equiv \operatorname{Null}(F)^\perp \), \( H_2 \equiv \operatorname{Null}(F) \), \( B_1 \equiv \operatorname{Range}(T) = T(H_1) \), and \( B_2 \equiv \operatorname{Range}(T)^\perp = \operatorname{Null}(T^*) \). Then \( F \) decomposes into the block form:

\[
F = \begin{pmatrix} \tilde{F} & 0 \\ 0 & 0 \end{pmatrix} : H_1 \to B_1 \oplus B_2,
\]

where \( \tilde{F} = F|_{H_1} : H_1 \to B_1 \) is an invertible operator. Let \( Y_1 \equiv T(B_1) \) and \( Y_2 \equiv Y_1^\perp = T(B_1)^\perp \). Notice that \( Y_1 = T(B_1) = TQ(B_1) \), where \( Q : B \to B_1 \subset B \) is orthogonal projection onto \( B_1 \). Since \( B_1 \) is closed and \( B_2 \) is finite dimensional, \( Q \) is Fredholm. Hence \( TQ \) is Fredholm and \( Y_1 = TQ(B_1) \) is closed in \( Y \) and is of finite codimension. Using the above decompositions, we may write \( T \) in the block form:

\[
T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : B_1 \oplus B_2 \to Y_1 \oplus Y_2.
\]

Since \( R = \begin{pmatrix} 0 & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : B \to Y \) is a finite rank operator and hence \( RF : H \to Y \) is finite rank, \( \text{index}(T - R) = \text{index}(T) \) and \( \text{index}(TF - RF) = \text{index}(TF) \). Hence without loss of generality we may assume that \( T \) has the form \( T = \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix} \), \( \tilde{T} = T_{11} \) and hence

\[
TF = \begin{pmatrix} \tilde{T} \tilde{F} & 0 \\ 0 & 0 \end{pmatrix} : H_1 \oplus H_2 \to Y_1 \oplus Y_2.
\]

We now compute the index \( T \). Notice that \( \text{Null}(T) = \text{Null}(\tilde{T}) \oplus B_2 \) and \( \text{Range}(T) = \tilde{T}(B_1) = Y_1 \). So

\[
\text{index}(T) = \text{index}(\tilde{T}) + \dim(B_2) - \dim(Y_2).
\]

Similarly,

\[
\text{index}(TF) = \text{index}(\tilde{T} \tilde{F}) + \dim(H_2) - \dim(Y_2),
\]

and as we have already seen

\[
\text{index}(F) = \dim(H_2) - \dim(B_2).
\]
Therefore,

\[
\text{index}(TF) - \text{index}(T) \text{ } - \text{index}(F) = \text{index}(\tilde{F}) - \text{index}(\tilde{T}).
\]

Since \(\tilde{F}\) is invertible, \(\text{Ran}(\tilde{T}) = \text{Ran}(\tilde{F})\) and \(\text{Nul}(\tilde{T}) \cong \text{Nul}(\tilde{F})\). Thus \(\text{index}(\tilde{F}) - \text{index}(\tilde{T}) = 0\) and the theorem is proved.

\(\blacksquare\)

35.6. **Tensor Product Spaces**. References for this section are Reed and Simon \(\text{[?]}\) (Volume 1, Chapter VI.5), Simon \(\text{[?]}\), and Schatten \(\text{[?]}\). See also Reed and Simon \(\text{[?]}\) (Volume 2 \(\S\) IX.4 and \(\S\)XIII.17).

Let \(H\) and \(K\) be separable Hilbert spaces and \(H \otimes K\) will denote the usual Hilbert completion of the algebraic tensors \(H \otimes_f K\). Recall that the inner product on \(H \otimes K\) is determined by \((h \otimes k, h' \otimes k') = (h, h')(k, k')\). The following proposition is well known.

**Proposition 35.29** (Structure of \(H \otimes K\)). There is a bounded linear map \(T : H \otimes K \rightarrow B(K, H)\) determined by

\[
T(h \otimes k)k' \equiv (k, k')h \text{ for all } k, k' \in K \text{ and } h \in H.
\]

Moreover \(T(H \otimes K) = HS(K, H)\) — the Hilbert Schmidt operators from \(K\) to \(H\). The map \(T : H \otimes K \rightarrow HS(K, H)\) is unitary equivalence of Hilbert spaces. Finally, any \(A \in H \otimes K\) may be expressed as

\[
A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n,
\]

where \(\{h_n\}\) and \(\{k_n\}\) are orthonormal sets in \(H\) and \(K\) respectively and \(\{\lambda_n\} \subset \mathbb{R}\) such that \(\|A\|^2 = \sum |\lambda_n|^2 < \infty\).

**Proof.** Let \(A \equiv \sum a_{ij} h_j \otimes k_i\), where \(\{h_i\}\) and \(\{k_j\}\) are orthonormal bases for \(H\) and \(K\) respectively and \(\{a_{ij}\} \subset \mathbb{R}\) such that \(\|A\|^2 = \sum |a_{ij}|^2 < \infty\). Then evidently,

\[
T(A)k \equiv \sum a_{ij} h_j(k_i, k) \leq \sum |a_{ij}|^2 (k_i, k)^2 \leq \sum |a_{ij}|^2 \|k\|^2.
\]

Thus \(T : H \otimes K \rightarrow B(K, H)\) is bounded. Moreover,

\[
\|T(A)\|_{HS}^2 \equiv \sum \|T(A)k_i\|^2 = \sum_{ij} |a_{ij}|^2 = \|A\|^2,
\]

which proves the \(T\) is an isometry.

We will now prove that \(T\) is surjective and at the same time prove Eq. (35.14). To motivate the construction, suppose that \(Q = T(A)\) where \(A\) is given as in Eq. (35.14). Then

\[
Q^*Q = T(\sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n)T(\sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n) = T(\sum_{n=1}^{\infty} \lambda_n^2 h_n \otimes k_n).
\]

That is \(\{k_n\}\) is an orthonormal basis for \((\text{null}Q^*)^\perp\) with \(Q^*Qk_n = \lambda_n^2 k_n\). Also \(Qk_n = \lambda_n h_n\), so that \(h_n = \lambda_n^{-1}Qk_n\).

We will now reverse the above argument. Let \(Q \in HS(K, H)\). Then \(Q^*Q\) is a self-adjoint compact operator on \(K\). Therefore there is an orthonormal basis \(\{k_n\}_{n=1}^{\infty}\)
for the \((\text{null}Q^*Q)^\perp\) which consists of eigenvectors of \(Q^*Q\). Let \(\lambda_n \in (0, \infty)\) such that \(Q^*Qk_n = \lambda_n^2k_n\) and set \(n_k = \lambda_n^{-1}Qk_n\). Notice that
\[
(h_n, h_m) = (\lambda_n^{-1}Qk_n, \lambda_m^{-1}Qk_m) = (\lambda_n^{-1}k_n, \lambda_m^{-1}Q^*Qk_m) = (\lambda_n^{-1}k_n, \lambda_m^{-1}k_m) = \delta_{mn},
\]
so that \(\{h_n\}\) is an orthonormal set in \(H\). Define
\[
A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n
\]
and notice that \(T(A)k_n = \lambda_n h_n = Qk_n\) for all \(n\) and \(T(A)k = 0\) for all \(k \in \text{null}Q = \text{null}Q^*Q\). That is \(T(A) = Q\). Therefore \(T\) is surjective and Eq. (35.14) holds. \(\blacksquare\)

Recall that \(\sqrt{1-z} = 1 - \sum_{i=1}^{\infty} c_i z^i\) for \(|z| < 1\), where \(c_i \geq 0\) and \(\sum_{i=1}^{\infty} c_i < \infty\). For an operator \(A\) on \(H\) such that \(A \geq 0\) and \(\|A\|_{B(H)} \leq 1\), the square root of \(A\) is given by
\[
\sqrt{A} = I - \sum_{i=1}^{\infty} c_i (A - I)^i.
\]
See Theorem VI.9 on p. 196 of Reed and Simon \[\text{[?]}. \] The next proposition is problem 14 and 15 on p. 217 of \[\text{[?]}. \] Let \(|A| \equiv \sqrt{A^*A}\).

**Proposition 35.30 (Square Root).** Suppose that \(A_n\) and \(A\) are positive operators on \(H\) and \(\|A-A_n\|_{B(H)} \to 0\) as \(n \to \infty\), then \(\sqrt{A_n} \to \sqrt{A}\) in \(B(H)\) also. Moreover, \(A_n\) and \(A\) are general bounded operators on \(H\) and \(A_n \to A\) in the operator norm then \(|A_n| \to |A|\).

**Proof.** Without loss of generality, assume that \(\|A_n\| \leq 1\) for all \(n\). This implies also that that \(\|A\| \leq 1\). Then
\[
\sqrt{A} - \sqrt{A_n} = \sum_{i=1}^{\infty} c_i \{(A_n - I)^i - (A - I)^i\}
\]
and hence
\[
(35.15) \quad \|\sqrt{A} - \sqrt{A_n}\| \leq \sum_{i=1}^{\infty} c_i \|(A_n - I)^i - (A - I)^i\|.
\]
For the moment we will make the additional assumption that \(A_n \geq \epsilon I\), where \(\epsilon \in (0, 1)\). Then \(0 \leq I - A_n \leq (1 - \epsilon)I\) and in particular \(\|I - A_n\|_{B(H)} \leq (1 - \epsilon)\).

Now suppose that \(Q, R, S, T\) are operators on \(H\), then \(QR - ST = (Q-S)R + S(R-T)\) and hence
\[
\|QR-ST\| \leq \|Q-S\|\|R\| + \|S\|\|R-T\|.
\]
Setting \(Q = A_n - I\), \(R \equiv (A_n - I)^{i-1}\), \(S \equiv (A - I)\) and \(T = (A - I)^{i-1}\) in this last inequality gives
\[
\|(A_n - I)^i - (A - I)^i\| \leq \|A_n - A\||(A_n - I)^{i-1}| + \|(A - I)|||A_n - I)^{i-1} - (A - I)^{i-1}|\|
\]
(35.16) \[\leq \|A_n - A\||(1-\epsilon)^{i-1} + (1-\epsilon)|||A_n - I)^{i-1} - (A - I)^{i-1}|.\]
It now follows by induction that
\[
\|\|(A_n - I)^i - (A - I)^i\| \leq i(1-\epsilon)^{i-1}\|A_n - A\|.
\]
Inserting this estimate into (35.15) shows that
\[
\|\sqrt{A} - \sqrt{A_n}\| \leq \sum_{i=1}^{\infty} c_i i(1-\epsilon)^{i-1}\|A_n - A\| = \frac{1}{2} \frac{1}{\sqrt{1-(1-\epsilon)}} \|A-A_n\| = \frac{1}{2} \frac{1}{\sqrt{\epsilon}} \|A-A_n\| \to 0.
\]
Therefore we have shown if \( A_n \geq \epsilon I \) for all \( n \) and \( A_n \to A \) in norm then \( \sqrt{A_n^* A} \to \sqrt{A} \) in norm.

For the general case where \( A_n \geq 0 \), we find that for all \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \sqrt{A_n + \epsilon} = \sqrt{A + \epsilon}.
\]

By the spectral theorem\(^{54}\)

\[
\|\sqrt{A + \epsilon - \sqrt{A}}\| \leq \max_{x \in \rho(A)} |\sqrt{x + \epsilon - \sqrt{x}}| \leq \max_{0 \leq x \leq \|A\|} |\sqrt{x + \epsilon - \sqrt{x}}| \to 0 \text{ as } \epsilon \to 0.
\]

Since the above estimates are uniform in \( A \geq 0 \) such that \( \|A\| \) is bounded, it is now an easy matter to conclude that Eq. (35.17) holds even when \( \epsilon = 0 \). □

Now suppose that \( A_n \to A \) in \( B(H) \) and \( A_n \) and \( A \) are general operators. Then \( A_n^* A_n \to A^*A \) in \( B(H) \). So by what we have already proved,

\[
|A_n| \equiv \sqrt{A_n^* A_n} \to |A| \equiv \sqrt{A^*A} \text{ in } B(H) \text{ as } n \to \infty.
\]

**Notation 35.31.** In the future we will identify \( A \in H \otimes K \) with \( T(A) \in HS(K, H) \) and drop \( T \) from the notation. So that with this notation we have \((h \otimes k)k' = (k, k')h\).

Let \( A \in H \otimes H \), we set \( \|A\|_1 \equiv \text{tr}\sqrt{A^*A} \equiv \text{tr}\sqrt{T(A)^* T(A)} \) and we let

\[
H \otimes_1 H \equiv \{ A \in H \otimes H : \|A\|_1 < \infty \}.
\]

We will now compute \( \|A\|_1 \) for \( A \in H \otimes H \) described as in Eq. (35.14). First notice that \( A^* = \sum_{n=1}^\infty \lambda_n k_n \otimes h_n \) and

\[
A^* A = \sum_{n=1}^\infty \lambda_n^2 k_n \otimes k_n.
\]

Hence \( \sqrt{A^* A} = \sum_{n=1}^\infty |\lambda_n| k_n \otimes k_n \) and hence \( \|A\|_1 = \sum_{n=1}^\infty |\lambda_n| \). Also notice that \( \|A\|^2 = \sum_{n=1}^\infty |\lambda_n|^2 \) and \( \|A\|_{op} = \max_n |\lambda_n| \). Since

\[
\|A\|_1^2 = \left\{ \sum_{n=1}^\infty |\lambda_n| \right\}^2 \geq \sum_{n=1}^\infty |\lambda_n|^2 = \|A\|^2,
\]

we have the following relations among the various norms,

\[
(35.18) \quad \|A\|_{op} \leq \|A\| \leq \|A\|_1.
\]

**Proposition 35.32.** There is a continuous linear map \( C : H \otimes_1 H \to \mathbb{R} \) such that \( C(h \otimes k) = (h, k) \) for all \( h, k \in H \). If \( A \in H \otimes_1 H \), then

\[
(35.19) \quad CA = \sum (e_m \otimes e_m, A),
\]

where \( \{e_m\} \) is any orthonormal basis for \( H \). Moreover, if \( A \in H \otimes_1 H \) is positive, i.e. \( T(A) \) is a non-negative operator, then \( \|A\|_1 = CA \).

\(^{54}\)It is possible to give a more elementary proof here. Indeed, assume further that \( \|A\| \leq \alpha < 1 \), then for \( \epsilon \in (0, 1 - \alpha) \), \( \|\sqrt{A + \epsilon - \sqrt{A}}\| \leq \sum_{i=1}^\infty c_i \|A + \epsilon - A^i\| \). But

\[
\|(A + \epsilon)^i - A^i\| \leq \sum_{k=1}^i \left( \frac{1}{k} \right) k^k \|A^{i-k}\| \leq \sum_{k=1}^i \left( \frac{1}{k} \right) k^k \|A\|^{i-k} = \left( \|A\| + \epsilon \right)^i - \|A\|^i,
\]

so that \( \|\sqrt{A + \epsilon - \sqrt{A}}\| \leq \sqrt{\|A\| + \epsilon - \sqrt{\|A\|}} \to 0 \) as \( \epsilon \to 0 \) uniformly in \( A \geq 0 \) such that \( \|A\| \leq \alpha < 1 \).
Proof. Let \( A \in H \otimes H \) be given as in Eq. (35.14) with \( \sum_{n=1}^{\infty} |\lambda_n| = \|A\|_1 < \infty \). Then define \( CA = \sum_{n=1}^{\infty} \lambda_n(h_n, k_n) \) and notice that \( |CA| \leq \sum |\lambda_n| = \|A\|_1 \), which shows that \( C \) is a contraction on \( H \otimes H \). (Using the universal property of \( H \otimes I H \) it is easily seen that \( C \) is well defined.) Also notice that for \( M \in \mathbb{Z}_+ \) that

\[
(35.20) \quad \sum_{m=1}^{M} (e_m \otimes e_m, A) = \sum_{n=1}^{\infty} \sum_{m=1}^{M} (e_m \otimes e_m, \lambda_n h_n \otimes k_n),
\]

\[
(35.21) \quad = \sum_{n=1}^{\infty} \lambda_n (P_M h_n, k_n),
\]

where \( P_M \) denotes orthogonal projection onto span\( \{e_m\}_{m=1}^{M} \). Since \( |\lambda_n(P_M h_n, k_n)| \leq |\lambda_n| \) and \( \sum_{n=1}^{\infty} |\lambda_n| = \|A\|_1 < \infty \), we may let \( M \to \infty \) in Eq. (35.21) to find that

\[
\sum_{n=1}^{\infty} (e_m \otimes e_m, A) = \sum_{n=1}^{\infty} \lambda_n (h_n, k_n) = CA.
\]

This proves Eq. (35.19).

For the final assertion, suppose that \( A \geq 0 \). Then there is an orthonormal basis \( \{k_n\}_{n=1}^{\infty} \) for the \( (\text{null} A) \perp \) which consists of eigenvectors of \( A \). That is \( A = \sum \lambda_n k_n \otimes k_n \) and \( \lambda_n \geq 0 \) for all \( n \). Thus \( CA = \sum \lambda_n \) and \( \|A\|_1 = \sum \lambda_n \).

Proposition 35.33 (Noncommutative Fatou’s Lemma). Let \( A_n \) be a sequence of positive operators on a Hilbert space \( H \) and \( A_n \to A \) weakly as \( n \to \infty \), then

\[
(35.22) \quad \text{tr} A \leq \liminf_{n \to \infty} \text{tr} A_n.
\]

Also if \( A_n \in H \otimes H \) and \( A_n \to A \) in \( B(H) \), then

\[
(35.23) \quad \|A\|_1 \leq \liminf_{n \to \infty} \|A_n\|_1.
\]

Proof. Let \( A_n \) be a sequence of positive operators on a Hilbert space \( H \) and \( A_n \to A \) weakly as \( n \to \infty \) and \( \{e_k\}_{k=1}^{\infty} \) be an orthonormal basis for \( H \). Then by Fatou’s lemma for sums,

\[
\text{tr} A = \sum_{k=1}^{\infty} (A e_k, e_k) = \sum_{k=1}^{\infty} \lim_{n \to \infty} (A_n e_k, e_k)
\]

\[
\leq \liminf_{n \to \infty} \sum_{k=1}^{\infty} (A_n e_k, e_k) = \liminf_{n \to \infty} \text{tr} A_n.
\]

Now suppose that \( A_n \in H \otimes H \) and \( A_n \to A \) in \( B(H) \). Then by Proposition 35.30, \( |A_n| \to |A| \) in \( B(H) \) as well. Hence by Eq. (35.22), \( \|A\|_1 \equiv \text{tr}|A| \leq \liminf_{n \to \infty} \text{tr}|A_n| \leq \liminf_{n \to \infty} \|A_n\|_1 \). ■

Proposition 35.34. Let \( X \) be a Banach space, \( B : H \times K \to X \) be a bounded bi-linear form, and \( \|B\| = \sup\{|B(h, k)| : \|h\| \|k\| \leq 1\} \). Then there is a unique bounded linear map \( \tilde{B} : H \otimes K \to X \) such that \( \tilde{B}(h \otimes k) = B(h, k) \). Moreover \( \|\tilde{B}\|_{op} = \|B\| \).
Then and hence as the proof on p. 299 of Reed and Simon. Suppose that Lemma 35.35.

\[ \sum_{n=1}^{\infty} |\lambda_n| |B(h_n, k_n)| \leq \sum_{n=1}^{\infty} |\lambda_n| \|B\| = \|A\|_1 \cdot \|B\|. \]

This shows that \( \tilde{B}(A) \) is well defined and that \( \|\tilde{B}\|_{op} \leq \|\tilde{B}\| \). The opposite inequality follows from the trivial computation:

\[ \|B\| = \sup\{|B(h, k)| : \|h\| \cdot |k| = 1\} = \sup\{|\tilde{B}(h \otimes k)| : \|h \otimes k\|_1 = 1\} \leq \|\tilde{B}\|_{op}. \]

\[ \text{Lemma 35.35. Suppose that } P \in B(H) \text{ and } Q \in B(K), \text{ then } P \otimes Q : H \otimes K \to H \otimes K \text{ is a bounded operator. Moreover, } P \otimes Q(H \otimes_1 K) \subset H \otimes_1 K \text{ and we have the norm equalities} \]

\[ \|P \otimes Q\|_{B(H \otimes K)} = \|P\|_{B(H)}\|Q\|_{B(K)} \]

and

\[ \|P \otimes Q\|_{B(H \otimes_1 K)} = \|P\|_{B(H)}\|Q\|_{B(K)}. \]

We will give essentially the same proof of \( \|P \otimes Q\|_{B(H \otimes K)} = \|P\|_{B(H)}\|Q\|_{B(K)} \) as the proof on p. 299 of Reed and Simon. Let \( A \in H \otimes K \) as in Eq. (35.14). Then

\[ (P \otimes I)A = \sum_{n=1}^{\infty} \lambda_n Ph_n \otimes k_n \]

and hence

\[ (P \otimes I)A((P \otimes I)A)^* = \sum_{n=1}^{\infty} \lambda_n^2 Ph_n \otimes Ph_n. \]

Therefore,

\[ \|(P \otimes I)A\|^2 = \text{tr}(P \otimes I)A((P \otimes I)A)^* \]

\[ = \sum_{n=1}^{\infty} \lambda_n^2 (Ph_n, Ph_n) \leq \|P\|^2 \sum_{n=1}^{\infty} \lambda_n^2 \]

\[ = \|P\|^2 \|A\|_1^2, \]

which shows that Thus \( \|P \otimes I\|_{B(H \otimes K)} \leq \|P\| \). By symmetry, \( \|I \otimes Q\|_{B(H \otimes K)} \leq \|Q\| \). Since \( P \otimes Q = (P \otimes I)(I \otimes Q) \), we have

\[ \|P \otimes Q\|_{B(H \otimes_1 K)} \leq \|P\|_{B(H)}\|Q\|_{B(K)}. \]

The reverse inequality is easily proved by considering \( P \otimes Q \) on elements of the form \( h \otimes k \in H \otimes K \).

\[ \text{Proof. Now suppose that } A \in H \otimes_1 K \text{ as in Eq. (35.14). Then} \]

\[ \|(P \otimes Q)A\|_1 \leq \sum_{n=1}^{\infty} |\lambda_n| \|Ph_n \otimes Qk_n\|_1 \leq \|P\|\|Q\| \sum_{n=1}^{\infty} |\lambda_n| = \|P\|\|Q\|\|A\|, \]

which shows that

\[ \|P \otimes Q\|_{B(H \otimes_1 K)} \leq \|P\|_{B(H)}\|Q\|_{B(K)}. \]

Again the reverse inequality is easily proved by considering \( P \otimes Q \) on elements of the form \( h \otimes k \in H \otimes_1 K \). \( \blacksquare \)
Lemma 35.36. Suppose that $P_m$ and $Q_m$ are orthogonal projections on $H$ and $K$ respectively which are strongly convergent to the identity on $H$ and $K$ respectively. Then $P_m \otimes Q_m : H \otimes_1 K \to H \otimes_1 K$ also converges strongly to the identity in $H \otimes_1 K$.

Proof. Let $A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n \in H \otimes_1 K$ as in Eq. (35.14). Then

$$
\|P_m \otimes Q_m A - A\|_1 \leq \sum_{n=1}^{\infty} |\lambda_n| \|P_m h_n \otimes Q_m k_n - h_n \otimes k_n\|_1
$$

$$
= \sum_{n=1}^{\infty} |\lambda_n| \|(P_m h_n - h_n) \otimes Q_m k_n + h_n \otimes (Q_m k_n - k_n)\|_1
$$

$$
\leq \sum_{n=1}^{\infty} |\lambda_n| \{\|P_m h_n - h_n\| \|Q_m k_n\| + \|h_n\| \|Q_m k_n - k_n\|\}
$$

$$
\leq \sum_{n=1}^{\infty} |\lambda_n| \{\|P_m h_n - h_n\| + \|Q_m k_n - k_n\|\} \to 0 \text{ as } m \to \infty
$$

by the dominated convergence theorem. ■