Problem 7.1.9

Let the vector from the center of the circle to the duck make an angle $\theta$ with the positive $x$-axis, let the vector from the dog to the duck make an angle $\phi$ with the vector from the duck to the center of the circle, and call the distance from the dog to the duck $R$, as shown below:

![Diagram](image)

Let the speed of the dog be $v$ (the length of the blue vector), and that of the duck be $\omega$ (the length of the green vector). Then $R$ is decreasing by $v$ and increasing by the projection of the duck’s velocity vector in the direction of the dog-duck vector: $\omega \cos(\frac{\pi}{2} - \phi) = \omega \sin \phi$. Thus

$$\dot{R} = -v + \omega \sin \phi.$$  

To understand how the angle $\phi$ is changing, extend the lines from the duck ($k$) to the center and the duck ($k$) to the dog ($g$) across the circle so that they become chords. Recall that the arc on the circle between the endpoints of these chords is $2\phi$. Consider the corresponding chords a moment later as the duck moves to a new position ($k'$), as shown in the next figure. The diameter is rotating at angular velocity $\omega$, so the arc on the circle between the endpoints of the chords is decreasing by $\omega$, but also increasing by the rate at which the chord through the dog is changing its endpoint ($l$). To see what this rate is, recall that the power of a point inside the circle is the product of the lengths into which it divides any chord. This means that the triangle $kk'g$ is similar to the triangle $ll'g$, and the ratio is $2 \cos \phi - R$ (the length of the chord $kl$ minus the length of the segment $kg$) to $R$ (the length of the segment $kg$). Thus the endpoint $l$ is moving at angular velocity
\[ \omega \left(2 \cos \phi - R\right)/R, \text{ since the endpoint } k \text{ is moving at angular velocity } \omega, \text{ and hence} \]

\[ 2 \dot{\phi} = -\omega + \frac{2 \cos \phi - R}{R} \omega \implies \dot{\phi} = \frac{\cos \phi - R}{R} \omega. \]