Please simplify your answers to the extent reasonable without a calculator, show your work, and explain your answers, concisely. If you set up an integral or a sum that you cannot evaluate, leave it as it is; and if the result is needed for the next part, say how you would use the result if you had it.

1. Suppose someone flips three fair coins and records the results. Let $H_1$ be the event that the first coin lands head up; let $M$ be the event that a majority of the coins land head up; and let $E$ be the event that an even number of the coins land head up.
   a. [4 points] Find $P(M)$ and $P(E)$.
      Let $H$ be the number of heads.
      Then $P(M) = P(H = 2) + P(H = 3) = (3 + 1)/8 = 1/2$
      and $P(E) = P(H = 0) + P(H = 2) = (1 + 3)/8 = 1/2$.
   b. [4 points] Find $P(M|H_1)$ and $P(E|H_1)$.
      Let $L$ be the number of heads in the last two flips.
      Then $P(M|H_1) = P(L = 2) + P(L = 1) = (1 + 2)/4 = 3/4$
      and $P(E|H_1) = P(L = 1) = 1/2$.
   c. [5 points] Which pairs of the three events $M$, $E$ and $H_1$ are independent?
      Since $P(M|H_1) \neq P(M)$, $M$ & $H_1$ are dependent. Since $P(E) = P(E|H_1)$, $E$ & $H_1$
      are independent. $P(M \cap E) = P(H = 2) = 3/8 \neq (1/2)(1/2) = P(M)P(E)$, so $M$
      & $E$ are dependent.

2. Alice is in 1st grade. At her school the 1st grade has 100 students divided into 4 equal sized classes. Assuming no student leaves the school, and no new students arrive, the same 100 students will be randomly divided into 4 equal sized classes in 2nd grade.
   a. [6 points] Some students may be in Alice’s class in both 1st and 2nd grade. What is
      the probability that there are exactly $k$ such students (not counting Alice)?
      This is an urn problem in which $k$ students are selected for Alice’s 2nd grade class
      from the 24 other students who were in Alice’s 1st grade class:
      $$P(k \text{ in both classes}) = \binom{24}{k} \binom{75}{24-k} \binom{99}{24}. $$
   b. [8 points] What is the expected number of students who are in Alice’s class both
      years (not counting Alice)?
      [Hint 1: Can you estimate what it is?]
      [Hint 2: You may find the following identity useful:
      $$\sum_{i=0}^{j} \binom{m}{i} \binom{n-m}{j-i} = \binom{n}{j}. $$
      This is the Vandermonde identity.]
Using Hint 1, since 24/99 of the other students were in Alice’s 1st grade class, we should expect 24 \cdot \frac{24}{99} \approx 6 students to be also in her class in 2nd grade. This suggests an easy way to calculate it: For each of the other 24 students in Alice’s 2nd grade class, let $X_i = 1$ if student $i$ was in Alice’s 1st grade class. Notice that $P(X_i = 1) = \frac{24}{99}$. Then the number of students in Alice’s class both years is $K = X_1 + \cdots + X_{24}$, so $E[K] = E[X_1] + \cdots + E[X_{24}] = 24 \cdot \frac{24}{99} = 64/11$.

A more complicated way to calculate it uses the answer to part (a) and Hint 2:

$$E[K] = \sum_{k=0}^{24} k \binom{24}{k} \binom{75}{24-k} = \left(\frac{99}{24}\right)^{-1} 24 \sum_{k=1}^{24} \left(\frac{23}{k-1}\right) \left(\frac{75}{24-k}\right).$$

Setting $i = k - 1$ and using the Vandermonde identity, we get

$$E[K] = \left(\frac{99}{24}\right)^{-1} 24 \sum_{i=0}^{23} \binom{23}{i} \binom{75}{23-i} = 24 \binom{98}{23} / \binom{99}{24} = \frac{64}{11}.$$

3. There are two bags, each containing what appear to be $20$ gold coins. One bag contains counterfeits, made badly, so that if you flip any one of them, the probability that it lands head up is 1/4. The other bag contains real gold coins, which have probability 1/2 of landing head up when flipped. After choosing one of the bags at random, a coin is drawn from it and flipped.

a. [3 points] What is the probability, $P(R)$, that the flipped coin is real?

$$P(R) = P(\text{pick bag with real coins}) = \frac{1}{2}.$$

b. [6 points] What is the probability, $P(H)$, that it lands head up?

$$P(H) = P(H|R)P(R) + P(H|\bar{R})P(\bar{R}) = (1/2)(1/2) + (1/4)(1/2) = 3/8.$$

c. [6 points] Suppose the coin lands head up. Now what is the probability, $P(R|H)$, that the coin is real?

$$P(R|H) = \frac{P(H|R)P(R)}{P(H)} = \frac{(1/2)(1/2)}{3/8} = \frac{2}{3}.$$

4. Let $X$ be a real random variable with probability density function

$$f_X(x) = \begin{cases} 1/x^2 & \text{if } x \geq 1; \\
0 & \text{otherwise}. \end{cases}$$

a. [4 points] Find the cumulative distribution function, $F_X(x)$, of $X$.

$$F_X(x) = \int_{-\infty}^{x} f_X(x')dx' = \int_{1}^{x} (x')^{-2}dx' = \left. -\frac{1}{x'} \right|_{1}^{x} = 1 - \frac{1}{x},$$

for $x \geq 1$, 0 otherwise.
b. [9 points] Suppose you have a uniform random variable, \( U \), with probability density function
\[
f_U(u) = \begin{cases} 
1 & \text{if } 0 < u < 1; \\
0 & \text{otherwise}.
\end{cases}
\]
How can you use \( U \) to simulate \( X \)? Your answer should include an explicit function \( g : (0, 1) \to (1, \infty) \) such that \( X = g(U) \).
Recall that \( F_X(X) \) is a uniform(0, 1) random variable, so just invert \( u = 1 - 1/x \) to get \( x = 1/(1 - u) \), i.e., \( g(u) = 1/(1 - u) \).

5. Let \( X \) and \( Y \) be random variables with joint probability density function
\[
f_{X,Y}(x, y) = \frac{1}{\pi \sqrt{3}} e^{-2(x^2 - xy + y^2)/3}.
\]

a. [7 points] What is \( f_X(x) \)?
\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy = \frac{1}{\pi \sqrt{3}} e^{-x^2/2} \int_{-\infty}^{\infty} e^{-2(y^2 - xy + x^2/4)/3}dy
\]
\[
= \frac{1}{\pi \sqrt{3}} e^{-x^2/2} \int_{-\infty}^{\infty} e^{-2(y-x/2)^2/(2\cdot3/4)}dy
\]
\[
= \frac{1}{\pi \sqrt{3}} e^{-x^2/2} \sqrt{2\pi \cdot 3/4} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

b. [3 points] What is \( \mathbb{E}[X] \)?
From (a), \( X \) is a standard normal random variable, so \( \mathbb{E}[X] = 0 \).

c. [7 points] What is \( \text{Cov}[X, Y] \)?
By symmetry, \( \mathbb{E}[Y] = 0 \) also, so
\[
\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY]
\]
\[
= \int_{-\infty}^\infty \int_{-\infty}^\infty xy f_{X,Y}(x, y)dx dy
\]
\[
= \frac{1}{\pi \sqrt{3}} \int_{-\infty}^\infty xe^{-x^2/2}dx \int_{-\infty}^\infty ye^{-(y-x/2)^2/(2\cdot3/4)}dy
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x \cdot \frac{x}{2} e^{-x^2/2}dx = \frac{1}{2} \text{Var}[X] = \frac{1}{2}.
\]
6. Choose two points uniformly at random in the interval \((0, 1)\).

a. [9 points] Find the probability density function for the distance between them.

Call the points \(X\) and \(Y\); a typical outcome is shown in the figure. The distance \(Z\) between them is \(|X - Y|\). The distribution function for \(Z\) is \(F_Z(z) = P(Z < z) = 1 - (1-z)^2\), twice the area between the red and blue lines, for \(0 < z < 1\). For \(z \leq 0\), \(F_Z(z) = 0\), and for \(1 \leq z\), \(F_Z(z) = 1\). Differentiating to get the probability density function gives:

\[
f_Z(z) = \begin{cases} 
2(1-z) & 0 < z < 1; \\
0 & \text{otherwise.}
\end{cases}
\]

b. [4 points] What is the average distance between the two points?

\[
E[Z] = \int_{0}^{1} z \cdot 2(1-z) \, dz = z^2 - \frac{2}{3} z^3 \bigg|_{0}^{1} = \frac{1}{3}.
\]

7. [15 points] Suppose two proof-readers independently read a long manuscript and find \(A\) and \(B\) mistakes, respectively. \(C\) mistakes are found by both proof-readers. Use the Law of Large Numbers to estimate how many mistakes in the manuscript have not been found.

[Hint: Let \(N\) be the total number of mistakes in the manuscript and assume \(A \sim \text{Binomial}(N, p)\) and \(B \sim \text{Binomial}(N, q)\).]

Using the Law of Large Numbers, \(A \approx pN\), \(B \approx qN\), and \(C \approx pqN\) (since the probability that a specific error is found by both proof-readers is \(pq\)). Then

\[
\frac{AB}{C} \approx \frac{pN \cdot qN}{pqN} = N,
\]

so the number of mistakes that have not been found is

\[
N - (A + B - C) \approx \frac{AB}{C} - (A + B - C).
\]
8. [Extra credit, 15 points] Let $S_1$ and $S_2$ be two spheres of radius 1 with centers a distance $r$ apart. Let $C_1$ and $C_2$ be great circles chosen independently, uniformly at random on $S_1$ and $S_2$, respectively. What is the probability that $C_1$ and $C_2$ are linked, as shown in the figure?

[Hint: Choosing a great circle uniformly at random is the same as choosing a unit vector $v$ (i.e., a point on the sphere) uniformly at random: the great circle is all the unit vectors $w$ that are perpendicular to $v$.]

If the circle on $S_2$ is reflected in the plane containing the intersection of the spheres, it becomes a great circle on $S_1$ and intersects $C_1$ in (generically) two points. One of these points lies inside $S_2$ if and only if $C_1$ and $C_2$ are linked. This point lies on the line in which the plane of $C_1$ and the plane of the reflected $C_2$ intersect. Each of these planes was chosen by choosing a uniformly random unit vector, so the line of intersection is also defined by a uniformly random unit vector. The probability that the line intersects $S_2$ is just twice the fractional area of the part of $S_1$ inside $S_2$, i.e., the area of the cap on $S_1$ beyond the plane in which $S_1$ and $S_2$ intersect, which is distance $r/2$ from the center. Recalling that this area is the same as the area of a unit radius cylinder with height $1 - r/2$, we obtain $P(C_1$ and $C_2$ are linked) $= 2(1 - r/2) \cdot 2\pi/4\pi = 1 - r/2$, for $0 \leq r < 2$, and 0 otherwise.