1.a. (20 points) Find the general solution to the system of equations:

\[
\frac{dx_1}{dt} = x_1 + x_2 \\
\frac{dx_2}{dt} = 4x_1 - 2x_2.
\]

Compute the eigenvalues and eigenvectors of the coefficient matrix:

\[
\begin{vmatrix}
1 - r & 1 \\
4 & -2 - r
\end{vmatrix} = r^2 + r - 6 = (r + 3)(r - 2) \implies r \in \{-3, 2\};
\]

\[
r = -3 \implies \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \implies \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}
\]

\[
r = 2 \implies \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \implies \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[\implies x = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}\]

b. (10 points) Sketch the solutions to this system of equations in the \((x_1, x_2)\) plane.

c. (10 points) Find the solution that goes through the point \((5, 0)\) at \(t = 0\).

\[
\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \implies x = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} - 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}\]
2. Consider the differential equation

\[(1 - x)\frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = g(x),\]

where \(g(x)\) is an arbitrary function and \(0 < x < 1\).

a. (5 points) Show that \(y_1(x) = e^x\) and \(y_2(x) = x\) solve this equation when \(g(x) = 0\).

\[(1 - x)e^x + xe^x - e^x = 0 \quad \implies \quad y_1(x) = e^x \text{ solves the equation}\]
\[(1 - x) \cdot 0 + x \cdot 1 - x = 0 \quad \implies \quad y_2(x) = x \text{ solves the equation}\]

b. (5 points) Show that the functions \(y_1(x)\) and \(y_2(x)\) are linearly independent on the interval \(0 < x < 1\). Compute the Wronskian:

\[W[y_1, y_2](x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1 - x) \neq 0 \text{ for } 0 < x < 1,\]

so these are linearly independent solutions.

c. (20 points) Find the general solution to this equation for an arbitrary function \(g(x)\). Hint: Your answer should involve integrals that depend upon \(g(x)\), which means that you won’t be able to evaluate them since you don’t know what \(g(x)\) is. Use variation of parameters:

\[y = u(x)e^x + v(x)x; \quad y' = u'e^x + v'x + uexe^x + v; \quad y'' = u'e^x + v' + uex\]

\[\implies (1 - x)(u'e^x + v' + uex) + x(uexe^x + v) - (u(x)e^x + v(x)x) = g(x)\]

\[\implies u'e^x + v' = g(x)/(1 - x)\]

Solving for \(u'\) and \(v'\) gives:

\[u' = -\frac{x e^{-x} g(x)}{(1 - x)^2} \quad \implies \quad u(x) = -\int \frac{xe^{-t}g(t)}{(1 - t)^2} dt\]

\[v' = \frac{g(x)}{(1 - x)^2} \quad \implies \quad v(x) = \int \frac{g(t)}{(1 - t)^2} dt\]

\[\implies y(x) = -e^x \int xe^{-t}g(t)/(1 - t)^2 dt + x \int \frac{g(t)}{(1 - t)^2} dt\]
3.a. (15 points) Find the general solution to the equation $y''' + 3y'' + 3y' + y = 0$. Try $y(t) = e^{rt}$. Plugging in gives:

$$r^3 e^{rt} + 3r^2 e^{rt} + 3re^{rt} + e^{rt} = 0 \implies 0 = r^3 + 3r^2 + 3r + 1 = (r + 1)^3.$$ 

By analogy with repeated roots for second order equations, this implies that the general solution is $y(t) = (c_1 + c_2 t + c_3 t^2)e^{-t}$.

b. (15 points) Define a change of variables so that this third order equation is equivalent to a system of 3 first order linear, constant coefficient ODEs, and write down that system of equations.

$$x_1 = y \quad \Rightarrow \quad x_1' = x_2$$
$$x_2 = y' \quad \Rightarrow \quad x_2' = x_3$$
$$x_3 = y'' \quad \Rightarrow \quad x_3' = -3x_3 - 3x_2 - x_1$$
4. (Extra credit: 15 points) Find the general solution of the second order differential equation
\[ x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} - 3y = \ln x \]
by making the change of variables \( x = e^t \) (which leads to a constant coefficient second order linear ODE).

\[
\frac{dy}{dx} = \frac{dt \ dy}{dx \ dt} = e^{-t} \frac{dy}{dt}
\]
\[
\frac{d^2 y}{dx^2} = \frac{dt \ d}{dx \ dt} \left( e^{-t} \frac{dy}{dt} \right) = e^{-t} \left( e^{-t} \frac{d^2 y}{dt^2} - e^{-t} \frac{dy}{dt} \right) = e^{-2t} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)
\]
\[\Rightarrow e^{2t} e^{-2t} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + 3e^{t} e^{-t} \frac{dy}{dt} - 3y = t \quad \Rightarrow \quad \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = t
\]

This is a constant coefficient inhomogeneous linear equation for \( y(t) \), so first solve the homogeneous equation using the characteristic equation:

\[ 0 = r^2 + 2r - 3 = (r + 3)(r - 1) \quad \Rightarrow \quad y_h(t) = c_1 e^{-3t} + c_2 e^{t}.
\]

Now try \( y_p(t) = at + b \), which implies \( t = 2a - 3(at + b) = -3at + (2a - 3b) \), so \( a = -1/3 \) and \( b = -2/9 \). Thus

\[
y(t) = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{3} t - \frac{2}{9}
\]
\[\Rightarrow y(x) = c_1 x^{-3} + c_2 x - \frac{1}{3} \ln x - \frac{2}{9}.
\]