2.a. The characteristic equation for the homogeneous equation is \( r^2 + 1 = 0 \), so \( r = \pm i \).

Thus the general solution to the homogeneous equation is:

\[ y(t) = c_1 \cos t + c_2 \sin t. \]

When \( b \neq 1 \), the inhomogeneous term \( \cos(bt) \) is not part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form:

\[ y = A \cos(bt) + B \sin(bt) \]

\[ \Rightarrow y' = -Ab \sin(bt) + Bb \cos(bt) \]

\[ \Rightarrow y'' = -Ab^2 \cos(bt) - Bb^2 \sin(bt). \]

Plugging these into the inhomogenous equation gives:

\[ -Ab^2 \cos(bt) - Bb^2 \sin(bt) + A \cos(bt) + B \sin(bt) = \cos(bt). \]

Since this must hold for all \( t \), it must in particular hold for \( t = 0 \) and \( t = \pi/(2b) \). At these values we get

\[ -Ab^2 + A = 1 \Rightarrow A = \frac{1}{1 - b^2}; \]

\[ -Bb^2 + B = 0 \Rightarrow B = 0. \]

So the general solution to the inhomogeneous equation is

\[ y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{1 - b^2} \cos(bt). \]

Then the initial conditions imply that

\[ 1 = y(0) = c_1 + \frac{1}{1 - b^2} \Rightarrow c_1 = \frac{-b^2}{1 - b^2} \]

and \( 0 = y'(0) = c_2 \). So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

\[ y(t) = \frac{-b^2}{1 - b^2} \cos t + \frac{1}{1 - b^2} \cos(bt). \]

b. When \( b = 1 \), the inhomogeneous term is part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form:

\[ y = t(A \cos t + B \sin t) \]

\[ \Rightarrow y' = t(-A \sin t + B \cos t) + (A \cos t + B \sin t) \]

\[ \Rightarrow y'' = t(-A \cos t - B \sin t) + 2(-A \sin t + B \cos t). \]

Plugging these into the inhomogenous equation gives:

\[ t(-A \cos t - B \sin t) + 2(-A \sin t + B \cos t) + t(A \cos t + B \sin t) = \cos t, \]
which simplifies to $2(-A\sin t + B\cos t) = \cos t$. Thus $A = 0$ and $B = 1/2$, so the general solution to the inhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t.$$  

Then the initial conditions imply that $1 = y(0) = c_1$ and $0 = y'(0) = c_2$. So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

$$y(t) = \cos t + \frac{1}{2} t \sin t.$$ 

(c. In the following plot $b = 1/2$ is blue, $b = 3/4$ is green, $b = 7/8$ is yellow, and $b = 1$ is red.}