MATH 102 - SOLUTIONS TO PRACTICE PROBLEMS - MIDTERM I

1. We carry out row reduction. We begin with the row operations

\[ R_2 \rightarrow R_2 - 2R_1, \; R_3 \rightarrow R_3 - R_1 \]

yielding the matrix

\[
\begin{bmatrix}
3 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]

This is already upper triangular hence

\[ U = \begin{bmatrix}
3 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix} \]

The lower triangular matrix equals

\[ L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} \]

2. For further reference, we begin by row-reducing the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & b_1 \\
2 & 3 & 5 & 7 & b_2 \\
-1 & 0 & -1 & -2 & b_3
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 & b_1 \\
0 & -1 & -1 & -1 & b_2 - 2b_1 \\
0 & 2 & 2 & 2 & b_1 + b_3
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 1 & 2 & 2b_2 - 3b_1 \\
0 & 1 & 1 & 1 & -b_2 + 2b_1 \\
0 & 0 & 0 & 0 & 2b_2 + b_3 - 3b_1
\end{bmatrix} \]

(i) The null space of \( A \) is also the null space of the row-reduced matrix

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The first two variables \( x, y \) are pivots, the last two variables \( z, w \) are free. We obtain the system

\[
x + z + 2w = 0 \implies x = -z - 2w
\]
\[
y + z + w = 0 \implies y = -z - w.
\]

We conclude

\[
\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} = z \begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix} + w \begin{bmatrix}
-2 \\
-1 \\
0 \\
1
\end{bmatrix}.
\]

Therefore \( N(A) \) has the basis

\[
N(A) = \text{span} \left\{ \begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-2 \\
-1 \\
0 \\
1
\end{bmatrix} \right\}.
\]

The nullity of \( A \) equals 2.
(ii) Each vector in the null space gives a relation between the columns of $A$. For instance, the vector
\[
\begin{bmatrix}
-1 \\
-1 \\
1 \\
0
\end{bmatrix} \in N(A) \implies -c_1 - c_2 + c_3 = 0
\]
and
\[
\begin{bmatrix}
-2 \\
-1 \\
0 \\
1
\end{bmatrix} \in N(A) \implies -2c_1 - c_2 + c_4 = 0.
\]

(iii) The last row of zeros in the row-reduced augmented matrix gives the equation
\[2b_2 + b_3 - 3b_1 = 0\]
that must be satisfied by all vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the column space.

(iv) The pivot columns, namely the first and second columns of $A$, give a basis for $C(A)$:
\[
C(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}.
\]
The rank of $A$ equals 2.

(v) The pivots are in the first and second row, so a basis for the row space of $A$ is
\[
\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

(vi) The dimension of the left-null space of $A$ is the number of rows minus the rank which is $3 - 2 = 1$.

(vii) We can easily find a particular solution
\[
x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\]
The general solution is $x = x_p + x_h$ hence
\[
x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.
\]

(viii) The rank of $A$ equals 2 which not equal to the number of rows or columns of $A$. Hence $A$ does not admit a left inverse nor a right inverse.
3. Assume first that the vectors \( \{u, v - w, u + v - 2w\} \) are linearly dependent. There exist constants \( a, b, c \) not all zero such that

\[
a u + b (v - w) + c (u + v - 2w) = 0.
\]

Rearranging, we obtain

\[
(a + c) u + (b + c) v + (b - 2c) w = 0.
\]

Since \( \{u, v, w\} \) is a basis, the vectors \( \{u, v, w\} \) must be independent. This implies that

\[
a + c = 0, \quad b + c = 0, \quad b - 2c = 0.
\]

Solving for \( a, b, c \) we find

\[
a = b = c = 0
\]

which is impossible by assumption that not all \( (a, b, c) \) are zero.

It follows that \( \{u, v - w, u + v - 2w\} \) are three linearly independent vectors in \( \mathbb{R}^3 \) hence they must form a basis of \( \mathbb{R}^3 \).

4.

(i) Not a vector space. For instance \( (1, 0, \ldots, 0) \) is in the set, but twice the vector \( 2(1, 0, \ldots, 0) \) is not in the set.

(ii) Vector space. This is the null space of the matrix

\[
\begin{bmatrix}
1 & -2 & 3 & -4 \\
1 & -1 & -1 & 1
\end{bmatrix}.
\]

Null spaces are always vector spaces.

(iii) Vector space. This is a bit harder to see. First, we determine the set a bit more explicitly.

The square of the distance to \( (3, -4, 0, 0) \) equals

\[
(x - 3)^2 + (y + 4)^2 + z^2 + w^2
\]

while the square distance to \( (0, -3, 4, 0) \) equals

\[
x^2 + (y + 3)^2 + (z - 4)^2 + w^2.
\]

Since the two distances must be equal, we have

\[
(x - 3)^2 + (y + 4)^2 + z^2 + w^2 = x^2 + (y + 3)^2 + (z - 4)^2 + w^2.
\]

Expanding out and canceling, we obtain the linear equation

\[-3x + y + 4z = 0.
\]

The requirement that the distances from \( (0, -3, 4, 0) \) and \( (0, 4, 3, 0) \) be equal can be worked out in a similar fashion. We obtain the equation

\[-7y + z = 0.
\]
Thus the set in question can be described by the two linear equations

\[-3x + y + 4z = 0, \quad -7y + z = 0.\]

This is a null space of the matrix

\[
\begin{bmatrix}
-3 & 1 & 4 & 0 \\
0 & -7 & 1 & 0
\end{bmatrix}
\]

hence it is a subspace.

(iv) Vector space. Indeed, if \(P\) and \(Q\) are two polynomials with

\[P(0) = P'(0) = P''(0) = 0, \quad Q(0) = Q'(0) = Q''(0) = 0\]

then their sum \(P + Q\) also satisfies

\[(P + Q)(0) = (P + Q)'(0) = (P + Q)''(0) = 0.\]

Similarly, scalar multiplication is preserved since \(cP\) satisfies

\[(cP)(0) = (cP)'(0) = (cP)''(0) = 0.\]

(v) Vector space. This is the set of matrices

\[A^T = -A.\]

This is a subspace since if \(A\) is skew symmetric so is \(cA\) since

\[(cA)^T = cA^T = -cA.\]

Similarly, if \(A, B\) are skew symmetric so is their sum \(A + B\) since

\[(A + B)^T = A^T + B^T = -A - B = -(A + B).\]

(vi) Not a vector space. If \(y\) is a solution of the equation, then \(y'' + 4y = \sin t.\) But \(2y\) is not a solution since

\[(2y)'' + 4(2y) = 2\sin t.\]

Since the equation changes, the solution set is not a vector space.

5. We calculate the null space of \(\text{rref}(A)\). This is the space spanned by \(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\). Elements in the null space of \(A\) give relations between the columns of \(A\). Writing \(v_1, v_2, v_3, v_4\) for the columns of \(A\) we must have

\[v_2 = v_4 - v_1 - v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}.\]

6. Note that

\[
\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.\]
Therefore,
\[
T \left( \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) = 2T \left( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) - T \left( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) + T \left( \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) = 2 \cdot \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.
\]

7.

(i) Clearly, \(\{1, x, x^2\}\) is a basis of \(P_2\) hence
\[
\dim P_2 = 3.
\]
To show that \(\{1, x - 1, x^2\}\) is also a basis, it suffices to prove that the three vectors are linearly dependent. Assume otherwise, so that
\[
a + b(x - 1) + cx^2 = 0
\]
for some constants \(a, b, c\). This rewrites as
\[
(a - b) + bx + cx^2 = 0 \implies a - b = 0, b = 0, c = 0 \implies a = b = c = 0
\]
proving the independence.

(ii) We calculate
\[
T(1) = 1, T(x - 1) = (x - 1), T(x^2) = 2x + x^2 = x^2 + 2(x - 1) + 2 \cdot 1.
\]
For the last equality, we needed to express \(T(x^2) = 2x + x^2\) in terms of the basis elements \(1, x - 1, x^2\). The matrix of \(T\) in this basis is given by collecting the coefficients of the basis elements \(1, x - 1, x^2\), yielding the matrix
\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}.
\]

8. True. To explain this fact, note that \(C\) has fewer rows than columns. Then \(C\) must have free variables. This in turn means that the null space \(N(C)\) contains a non-zero vector \(x\):
\[
Cx = 0.
\]
We claim that \(x\) is also in the null space of \(A\). Indeed,
\[
Ax = BCx = B \cdot 0 = 0.
\]
This shows that \(N(A)\) contains a nonzero vector.

9. Consider \(\text{Mat}_2\) the space of \(2 \times 2\) matrices. This is a 4-dimensional vector space. The 5 matrices \(I, A, A^2, A^3, A^4\) are vectors in \(\text{Mat}_2\). Since 5 vectors in a 4-dimensional vector space must be linearly dependent, we can therefore find constants \(c_0, c_1, c_2, c_3, c_4\) not all zero such that
\[
c_0I + c_1A + c_2A^2 + c_3A^3 + c_4A^4 = 0.
\]