Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [5+5 points] Let \( R \) be a ring. (a) Carefully state what it means for \( R \) to be a “domain” and (b) give an example with justification of a ring which is not a domain.

Problem 2: [15 points] Prove that \( \log_3(7) \) is not a rational number.

Problem 3: [15 points] Let \( \mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\} \) be the integer lattice in the Euclidean plane \( \mathbb{R}^2 \). Let \( \ell_1 \) and \( \ell_2 \) be the lines with equations \( 30x + 12y = 9 \) and \( 30x + 12y = 42 \). Does \( \ell_1 \) contain any points in \( \mathbb{Z}^2 \)? Does \( \ell_2 \)? Justify your answers.

Problem 4: [10 + 10 points] Let \( a, b \in \mathbb{Z} \) be given by \( a = 210 \) and \( b = 45 \). (a) Find the greatest common divisor \((a, b)\) and (b) express \((a, b)\) as a linear combination of \( a \) and \( b \) with integer coefficients.

Problem 5: [5+15 points] Let \( D \) be a domain. (a) Carefully state what it means for \( D \) to be a “unique factorization domain (UFD)” (you may take the notions of ‘prime’ and ‘associate’ for granted). (b) Let \( \mathbb{Z}[2i] = \{a + 2ib : a, b \in \mathbb{Z}\} \). You may assume without proof that \( \mathbb{Z}[2i] \) is a domain. Prove that \( \mathbb{Z}[2i] \) is not a UFD. (Hint: 4.)

Problem 6: [15 points] Recall that for \( n \in \mathbb{Z}^+ \), \( n! := n(n - 1)(n - 2) \cdots 1 \). How many zeros are there at the end of the decimal expansion of \((100!)\)?