Math 140A: Final Exam  
Foundations of Real Analysis

- You have 3 hours.
- No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- You may not use homework problems (without proof) in your solutions.
1. (10 points) Let $J$ be the set of all positive integers. Let $A$ be an infinite set.
   (a) (5 points) Prove that there exists a 1-1 function $f : J \rightarrow A$.
   (b) (5 points) Prove that the set of all 1-1 functions $f : J \rightarrow A$ is uncountable.
2. (10 points) Let $X$ be a nonempty set. For $x \in X$ and $y \in X$, define

$$d(x, y) = \begin{cases} 
1, & \text{if } x \neq y \\
0, & \text{if } x = y 
\end{cases}$$

(a) (3 points) Prove that $d$ is a distance function.

(b) (3 points) Prove that if $X$ is connected, then $X$ has exactly one element.

(c) (4 points) Prove that if $X$ is compact, then $X$ is finite.
3. (10 points) Let \( \{x_n\} \) be a sequence of real numbers. Assume that the “even” and “odd” subsequences \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) are convergent. Denote \( a = \lim_{n \to \infty} x_{2n} \) and \( b = \lim_{n \to \infty} x_{2n+1} \).

(a) (5 points) Prove that if \( a \neq b \), then the sequence \( \{x_n\} \) is not convergent.

(b) (5 points) Prove that if \( a = b \), then the sequence \( \{x_n\} \) is convergent and \( \lim_{n \to \infty} x_n = a \).
4. (10 points) Let \( \{x_n\} \) be a bounded sequence of real numbers. Denote \( \alpha = \limsup_{n \to \infty} x_n \).

Define a new sequence \( \{y_m\} \) by letting \( y_m = \sup\{x_n | n \geq m\} \), for every \( m \geq 1 \).

(a) (5 points) Prove that the sequence \( \{y_m\} \) is monotonically decreasing and convergent.

(b) (5 points) Prove that \( \lim_{m \to \infty} y_m = \alpha \).
5. (10 points)

(a) (5 points) Prove that the series $\sum \frac{n^3}{3^n}$ converges.

(b) (5 points) Let $\{a_n\}$ be a sequence of real numbers such that $a_1 \geq a_2 \geq a_3 \geq ... \geq 0$. Assume that $3a_{2n} \leq a_n$, for all $n \geq 1$. Prove that the series $\sum a_n$ converges.
6. (10 points) Let \( \{a_n\} \) be a sequence of real numbers.

(a) (5 points) Assume that the series \( \sum a_n \) is absolutely convergent. Let \( \{b_n\} \) be a bounded sequence of real numbers. Prove that the series \( \sum a_n b_n \) is absolutely convergent.

(b) (5 points) Assume that the series \( \sum a_n b_n \) is convergent, for any bounded sequence \( \{b_n\} \) of real numbers. Prove that the series \( \sum a_n \) is absolutely convergent.
7. Let $A$ be a nonempty set of real numbers and let $f : A \to [0, \infty)$ be given by $f(x) = x^2$.

(a) (5 points) Prove that if $A$ is bounded, then $f$ is uniformly continuous.

(b) (5 points) Prove that if $A$ is open and $f$ is uniformly continuous, then $A$ is bounded.
8. Let \( \mathbb{R} \) denote the set of real numbers. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be continuous functions. Assume that \( f(x) = g(x) \), for every rational number \( x \).
Prove that \( f(x) = g(x) \), for every \( x \in \mathbb{R} \).
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