Problem 1.

Let \( \{a_n\}_{n \geq 1} \) be an increasing sequence of real numbers.

(i) If \( \{a_n\} \) has a bounded subsequence, show that \( \{a_n\} \) is itself bounded.

(ii) If \( \{a_n\} \) has a convergent subsequence, show that \( \{a_n\} \) is itself convergent.

Solution:

(i) Clearly, \( \{a_n\} \) is bounded from below by \( a_1 \). We show that \( \{a_n\} \) is bounded above. Assume that the subsequence \( \{a_{n_k}\} \) is bounded above by \( M \). We show that \( a_m \leq M \) for all \( m \), completing the proof. Fix an index \( m \). Since \( n_k \to \infty \), we can find \( k \) sufficiently large such that \( m < n_k \). Then

\[
a_m \leq a_{n_k} \leq M,
\]

as claimed.

(ii) If \( \{a_{n_k}\} \) is a convergent subsequence, then \( \{a_{n_k}\} \) is bounded. By part (i), it follows that \( \{a_n\} \) is bounded. But bounded and increasing implies convergent as shown in class.
Problem 2.

For each of the series below indicate whether it either:

(a) converges absolutely;
(b) converges but not absolutely;
(c) does not converge.

Please briefly justify your answer.

(i)

\[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \]

(ii)

\[ \sum_{n=1}^{\infty} (-1)^n \frac{2n - 1}{3n + 2} \]

(iii)

\[ \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{3} + \sin \frac{\pi n}{4}}{\sqrt{n} + 2^n} \]

Solution:

(i) The series converges but not absolutely. Let \( a_n = \frac{1}{\sqrt{n}} \). To see convergence of

\[ \sum_{n=1}^{\infty} (-1)^n a_n, \]

we use the Abel test for the decreasing sequence \( a_n \to 0 \). To see that the convergence is not absolute, note that

\[ \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \]

diverges as shown in class.

(ii) The series diverges. Let \( a_n = \frac{2n - 1}{3n + 2} \). If \( \sum_{n=1}^{\infty} (-1)^n a_n \) were to converge, then \( a_n \to 0 \) as \( n \to \infty \). But \( a_n \to \frac{2}{3} \), contradiction.

(iii) The series converges absolutely. Indeed, let

\[ a_n = \frac{\sin \frac{\pi n}{3} + \sin \frac{\pi n}{4}}{\sqrt{n} + 2^n}. \]

We have

\[ |a_n| \leq \frac{|\sin \frac{\pi n}{3}| + |\sin \frac{\pi n}{4}|}{\sqrt{n} + 2^n} \leq \frac{2}{\sqrt{n} + 2^n} \leq \frac{2}{2^n}. \]

Since the series \( \sum_{n=1}^{\infty} \frac{2}{2^n} \) converges, the series \( \sum_{n=1}^{\infty} |a_n| \) also converges by the comparison test.
Problem 3.

Let $(X,d)$ be a metric space. A subset $A \subset X$ is said to be Lindelöf if every cover of $A$ by open sets of $X$ has an at most countable subcover. In particular, any compact set is Lindelöf.

If $f : X \to Y$ is a continuous map and $A$ is Lindelöf subset of $X$, then $f(A)$ is Lindelöf subset of $Y$.

**Solution:** Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a cover of $f(A)$ by open sets in $Y$ indexed by a set $I$. Then

$$f(A) \subset \bigcup_{\alpha \in I} U_\alpha \implies A \subset \bigcup_{\alpha \in I} f^{-1}(U_\alpha).$$

Since $f$ is continuous and $U_\alpha$ is open in $Y$, it follows that $f^{-1}(U_\alpha)$ is open in $X$. Since $A$ is Lindelöf, we can find a subcover indexed by an at most countable set $J$:

$$A \subset \bigcup_{\alpha \in J} f^{-1}(U_\alpha).$$

It follows that

$$f(A) \subset \bigcup_{\alpha \in J} U_\alpha.$$

Since we have found an at most countable subcover of $\mathcal{U}$, we established that $f(A)$ is Lindelöf.
Problem 4.

Let \( f, g : \mathbb{R} \to \mathbb{R} \) be uniformly continuous and bounded functions. Show that:

(i) For all \( x, y \in \mathbb{R} \), we have
\[
|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|.
\]
(ii) Use part (i) to show that product \( fg : \mathbb{R} \to \mathbb{R} \) is uniformly continuous.

Solution:

(i) We have
\[
|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|
\]
\[
= |f(x)(g(x) - g(y)) + (f(x) - f(y))g(y)|
\]
\[
\leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)|
\]
where in the last line we used the triangle inequality.

(ii) Assume that
\[
|f(x)| \leq M_1 \text{ and } |g(x)| \leq M_2
\]
for constants \( M_1, M_2 \). Let \( \epsilon > 0 \) be fixed. By uniform continuity for \( f \), we can find \( \delta_1 > 0 \) such that if
\[
|x - y| < \delta_1 \implies |f(x) - f(y)| < \frac{\epsilon}{M_1 + M_2}.
\]
Similarly, using the uniform continuity of \( g \), we can find \( \delta_2 > 0 \) such that if
\[
|x - y| < \delta_2 \implies |g(x) - g(y)| < \frac{\epsilon}{M_1 + M_2}.
\]
Let \( \delta = \min(\delta_1, \delta_2) > 0 \). For \( |x - y| < \delta \) the two inequalities above hold true. Using (i), we estimate for \( |x - y| < \delta \):
\[
|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| < M_1 \cdot \frac{\epsilon}{M_1 + M_2} + M_2 \cdot \frac{\epsilon}{M_1 + M_2} = \epsilon.
\]
This shows the product function \( fg \) is uniformly continuous.
Problem 5.

Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a function such that

$$d(f(x), f(y)) \leq \frac{1}{2} d(x, y), \quad \text{for all } x, y \in X.$$ 

Show that there exists $x^* \in X$ such that $f(x^*) = x^*$ by following the steps below:

(i) Pick $x_0 \in X$ arbitrary, and define the recursive sequence

$$x_{n+1} = f(x_n).$$

Use induction on $n$ to show that

$$d(x_{n+1}, x_n) \leq C \frac{2^n}{2^n}$$

for the constant $C = d(x_0, x_1)$ that may depend on $x_0$ and on the function $f$.

(ii) Use (i) to show that the sequence $\{x_n\}$ is Cauchy hence convergent.

(iii) Let $x^* = \lim_{n \to \infty} x_n$. Show that $f(x^*) = x^*$.

(iv) Show that the point $x^*$ with the property that $f(x^*) = x^*$ is unique.

Solution:

(i) We show by induction on $n$ that $d(x_{n+1}, x_n) \leq C \frac{2^n}{2^n}$. The case $n = 0$ follows since $C = d(x_0, x_1)$. For the inductive step, we assume

$$d(x_{n+1}, x_n) \leq C \frac{2^n}{2^n}$$

and compute

$$d(x_{n+2}, x_{n+1}) = d(f(x_{n+1}), f(x_n)) \leq \frac{1}{2} d(x_{n+1}, x_n) \leq \frac{1}{2} \cdot C \frac{2^n}{2^n} = C \frac{2^{n+1}}{2^n}.$$ 

(ii) We show that $\{x_n\}$ is Cauchy. We fix $\varepsilon > 0$. Since

$$\sum_{k=0}^{\infty} \frac{C}{2^k}$$

converges, we can find $M > 0$ such that for all $n, m \geq M$ we have

$$\sum_{k=n}^{m-1} \frac{C}{2^k} < \varepsilon.$$ 

But then for $n, m \geq M$, we have by the triangle inequality first, then by part (i) that

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \leq \frac{C}{2^n} + \ldots + \frac{C}{2^{m-1}} < \varepsilon.$$ 

This shows that $\{x_n\}$ is Cauchy. Since $X$ is complete, $\{x_n\}$ must converge.

(iii) The function $f$ is clearly uniformly continuous, being Lipschitz. If

$$x^* = \lim_{n \to \infty} x_n \implies f(x^*) = \lim_{n \to \infty} f(x_n)$$
using continuity. By uniqueness of limits,

\[ f(x) = \lim_{n \to \infty} x_{n+1} = x^*. \]

(iv) To show uniqueness of \( x^* \), assume that

\[ f(x^*) = x^* \text{ and } f(x^{**}) = x^{**}. \]

We show \( x^* = x^{**} \). By assumption

\[ d(f(x^*), f(x^{**})) \leq \frac{1}{2} d(x^*, x^{**}) \implies d(x^*, x^{**}) \leq \frac{1}{2} d(x^*, x^{**}) \implies d(x^*, x^{**}) \leq 0 \]

\[ \implies d(x^*, x^{**}) = 0 \implies x^* = x^{**} \]

which is what we wanted.
Problem 6.

The following four statements are all false. Provide counterexamples to each one of them. (The same function may be used as counterexample for several of these statements.)

(i) If \( f : X \to Y \) is continuous and \( U \subset X \) is open, then \( f(U) \) is open in \( Y \).
(ii) If \( f : X \to Y \) is continuous and \( F \subset X \) is closed, then \( f(F) \) is closed in \( Y \).
(iii) If \( f : X \to Y \) is continuous and \( K \subset Y \) is compact, then \( f^{-1}(K) \) is compact.
(iv) If \( f : X \to Y \) is continuous and \( C \subset Y \) is connected, then \( f^{-1}(C) \) is connected.

Solution:

(i) Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \). If \( U = (-1, 1) \) then \( f(U) = [0, 1) \) which is not open.

(ii) Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x^2 + 1} \). Then \( F = [0, \infty) \) is closed in \( \mathbb{R} \), but \( f(F) = (0, 1] \) is not closed in \( \mathbb{R} \).

(iii) Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x^2 + 1} \). Then \( K = [0, 1] \) is compact but \( f^{-1}(K) = \mathbb{R} \) since for all \( x \) we have \( 0 < f(x) \leq 1 \implies f(x) \in K \). Clearly \( f^{-1}(K) = \mathbb{R} \) is not compact.

(iv) Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \). Then \( C = \{1\} \) is connected, but \( f^{-1}(C) = \{-1, 1\} \) is not connected.
Problem 7.

Let $(X,d)$ be a metric space and let $K \subset X$, $p \in X \setminus K$. Define the function $f : K \to \mathbb{R}$ by

$$f(x) = d(x,p).$$

(i) Show the function $f$ is uniformly continuous.

(ii) If $K$ is compact, show that there exists $m, M > 0$ such that $m \leq d(x,p) \leq M$ for all $x \in K$.

(iii) If $K$ is only assumed to be closed, show that there exists $m > 0$ such that $m \leq d(x,p)$ for all $x \in K$.

Solution:

(i) From the triangle inequality, we have

$$d(x,p) \leq d(y,p) + d(x,y) \implies f(x) \leq f(y) + d(x,y) \implies f(x) - f(y) \leq d(x,y).$$

Similarly, exchanging the roles of $x$ and $y$,

$$f(y) - f(x) \leq d(x,y).$$

Thus

$$|f(x) - f(y)| \leq d(x,y).$$

If $\epsilon > 0$ is arbitrary, and $d(x,y) < \epsilon$, then

$$|f(x) - f(y)| < \epsilon$$

showing uniform continuity of $f$.

(ii) The continuous function $f$ must be bounded from below by $m$ on the compact $K$, and the minimum value $m$ is achieved at some $x_0 \in K$. Then

$$m = f(x_0) = d(p,x_0) \neq 0$$

since otherwise $p = x_0$, contradicting $x_0 \in K$ while $p \not\in K$. Thus $m > 0$. In other words, for all $x \in K$, we have

$$d(x,p) = f(x) \geq m.$$ 

Similarly, the continuous function $f$ is bounded from above by some $M$ on the compact $K$. Thus

$$m \leq d(x,p) \leq M.$$ 

(iii) Assume otherwise. Then for all $m > 0$, there exists $x(m) \in K$ such that $d(x(m),p) < m$.

For each natural number $n$, set $m = \frac{1}{n}$. In this case, we find $x_n \in K$ such that $d(x_n,p) < \frac{1}{n}$.

Therefore

$$\lim_{n \to \infty} x_n = p.$$ 

Since $x_n \in K$ and $K$ is closed, it follows that $p \in K$, a contradiction.
Problem 8.

Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function such that

\[
\lim_{x \to \infty} f(x) = 0.
\]

Show that \( f \) is uniformly continuous.

Solution: We fix \( \epsilon > 0 \). Since

\[
\lim_{x \to \infty} f(x) = 0
\]

we can find \( M > 0 \) such that if

(1) \( x \geq M \implies |f(x)| < \frac{\epsilon}{3} \).

Over the interval \([0, M]\) the function \( f \) is continuous, hence uniformly continuous, so we can find \( \delta > 0 \) such that

(2) \( \forall x, y \in [0, M], |x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{3} \).

We claim that if \( x, y \in \mathbb{R} \),

\[
|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.
\]

Indeed,

- the inequality is true for \( x, y \in [0, M] \) by (2);
- if \( x, y > M \) then by (1) and the triangle inequality, we find

\[
|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.
\]

- otherwise, let us assume \( x \in [0, M] \) but \( y \in [M, \infty) \). We find

\[
|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M)| + |f(y)|
\]

by the triangle inequality. Since \( x \leq M < y \) we have

\[
|x - M| \leq |x - y| < \delta \implies |f(x) - f(M)| < \frac{\epsilon}{3},
\]

by (2). But by (1), we have

\[
|f(M)| < \frac{\epsilon}{3}, \quad |f(y)| < \frac{\epsilon}{3},
\]

so that

\[
|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M)| + |f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

This shows that \( f \) is uniformly continuous.