1. 

(i) Write down a proof of Theorem 3.17(a) in your own words.
(ii) Prove that 
\[ \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \]
as long as the right hand side is well-defined i.e. it is not of the form \( \infty + (-\infty) \) or vice-versa.

It may be necessary here to use Theorem 3.17(a).
(iii) Give an example of sequences \( \{a_n\} \) and \( \{b_n\} \) for which equality does not hold in (ii).

2. Rudin, chapter 3, solve 6(abc), 7, 11, 14(abd).

3. Let \( \ell^2 \) denote the space of sequences \( \{x_n\} \) of real numbers such that the series 
\[ \sum_{n=1}^{\infty} x_n^2 \]
converges. The square root of the sum of the series, that is
\[ \|\{x_n\}\| := \left( \sum_{n=1}^{\infty} x_n^2 \right)^{\frac{1}{2}} \]
is called the \( \ell^2 \)-norm of the sequence. For two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( \ell^2 \), define 
\[ d(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\| \]
as the \( \ell^2 \)-norm of the difference sequence.

(i) Show that the distance is well-defined and that \((\ell^2, d)\) is a metric space. You may want to remember Cauchy-Schwarz.
(ii) For each \( m \geq 1 \), consider the sequence \( s_m \) whose terms are all equal to 0 except for the \( m^{th} \) term which is 1. That is,
\[ s_m = (0, 0, \ldots, 0, 1, 0, \ldots) \]
Show that for each \( m \geq 1 \), \( s_m \) is a member of \( \ell^2 \) and prove that the resulting sequence \( \{s_1, s_2, \ldots, s_m, \ldots\} \) is not Cauchy in \( \ell^2 \).
(iii) In the metric space \( \ell^2 \), consider the closed unit ball of center the zero sequence i.e. the set 
\[ K = \{ \text{sequences } \{x_n\} \text{ whose norm } \|\{x_n\}\| \leq 1 \} \]
Show that \( K \) is closed and bounded but not compact by exhibiting a sequence in \( K \) which does not have a convergent subsequence.
4. Let $q$ be a positive rational number. Show that

$$\lim_{n \to \infty} \left(1 + \frac{q}{n}\right)^n = e^q.$$ 

You may want to write $q = \frac{a}{b}$ and first consider the cases $a = 1$ and $b = 1$ as warm-ups.

5. Show that the series

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}$$

converges.

6. Assume that $\{a_n\}$ is a sequence of positive numbers such that $a_1 \geq a_2 \geq a_3 \geq \ldots$ and

$$3a_{2n} \leq a_n.$$ 

Show that $\sum_{n=1}^{\infty} a_n$ converges.