Math 140A - Homework 7. Due Friday, December 12.

1. Rudin, Chapter 4, solve problems 14, 16.

2. Show that if $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = -1$ and $f(1) = 0$, then there exists $x \in [0, 1]$ such that
   $$f(x) = 1 - 2x.$$

3. Show that the function
   $$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 
   0 & \text{if } x = 0 \\
   \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0
   \end{cases}$$
does have the intermediate value property without being continuous.

4. Solve the following version of Rudin, Chapter 4, Problem 18.

Consider the Thomae function $f : (0, 1) \to \mathbb{R}$ given by
   $$f(x) = \begin{cases} 
   \frac{1}{q} & \text{for rational numbers written in lowest terms } x = \frac{p}{q}, \gcd(p, q) = 1 \\
   0 & \text{if } x \text{ is irrational.}
   \end{cases}$$

(i) Explain that $f$ is discontinuous at all rational numbers $x$.

(ii) Show that for any $\epsilon > 0$, there are finitely many rational numbers $x \in [0, 1]$ such that $f(x) > \epsilon$.

   Hint: For instance, think of a very small $\epsilon$, such as $\epsilon = \frac{1}{10000000000}$. How many $x$’s have the property that $g(x) > \epsilon$? Can you then write down a proof that works for any $\epsilon$?

(iii) Using (iii) and the $\epsilon - \delta$ definition, show that $f$ continuous at all irrational numbers $x$ in $(0, 1)$.

(iv) What kind of discontinuities does $f$ have?