Math 140A: Midterm 1
Foundations of Real Analysis

• You have 50 minutes.
• No books and notes are allowed.
• You may quote any result stated in the textbook or in class.
• You may not use homework problems (without proof) in your solutions.
1. (10 points) Let $A$ and $B$ be two nonempty sets of positive real numbers. The “product of $A$ and $B$” is defined as $C = \{ab | a \in A, b \in B\}$.

Prove that $C$ is bounded below and $\inf C = (\inf A)(\inf B)$. 

2. (10 points)
(a) (5 points) Prove that \( \inf \left\{ \frac{1}{n} \mid n \text{ positive integer} \right\} = 0. \)
(b) (5 points) Prove that if \( x, y, z \in \mathbb{R}^k \) (the euclidean \( k \)-space), then
\[
|x| + |y| + |z| \leq |x + y - z| + |x - y + z| + |-x + y + z|.
\]
3. (10 points) Let $J$ be the set of all positive integers.
(a) (5 points) Let $A$ be the set of all finite subsets of $J$. Prove that $A$ is countable.
(b) (5 points) Let $B$ be the set of all subsets of $J$. Prove that $B$ is uncountable.
4. (10 points) Let $X$ be a metric space with distance function $d$. Let $A$ be a subset of $X$ and $x$ be a point in $X$. The “distance from $x$ to $A$” is defined as $d(x, A) = \inf \{d(x, y) \mid y \in A\}$.

(a) (5 points) Prove that $x \in \overline{A}$ if and only $d(x, A) = 0$.

(b) (5 points) Assume that $A$ is compact. Prove that there exists a point $y \in A$ such that $d(x, y) = d(x, A)$. 

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