Math 140A - Fall 2014 - Midterm I

Name: ____________________________

Student ID: ________________________

Instructions:

Please print your name, student ID.

During the test, you may not use books, calculators or telephones.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 10 points each. You have 50 minutes to complete the test.

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Problem 1. [10 points]

Let \((X, \rho)\) and \((Y, \tau)\) be metric spaces with metrics \(\rho\) and \(\tau\). Endow the product \(X \times Y\) with the distance function
\[
d((x, y), (x', y')) = \rho(x, x') + \tau(y, y') \quad \text{where} \quad (x, y) \in X \times Y, (x', y') \in X' \times Y'.
\]
You do not need to check that \(d\) is a distance function on the product \(X \times Y\).

(i) Show that if \(U\) is open in \(X\) and \(V\) is open in \(Y\) then \(U \times V\) is open in \(X \times Y\).

(ii) Show that if \(U\) is closed in \(X\) and \(V\) is closed in \(Y\) then \(U \times V\) is closed in \(X \times Y\).

Solution:

(i) Pick \((u, v) \in U \times V\). We show \((u, v)\) is an interior point of \(U \times V\). Since \(u \in U\) and \(U\) is open, there exists a ball in the \(\rho\)-metric such that \(B(u, r_1) \subset U\). Similarly, there exists a ball in the \(\tau\)-metric such that \(B(v, r_2) \subset V\). Let \(r = \min(r_1, r_2)\). We claim that the \(d\)-ball
\[
B((u, v), r) \subset U \times V.
\]
This would show \((u, v)\) is interior point. Indeed, if \((u', v') \in B((u, v), r)\) then
\[
d((u, v), (u', v')) < r \implies \rho(u, u') + \tau(v, v') < r \implies \rho(u, u') < r \leq r_1, \quad \tau(v, v') < r \leq r_2
\]
\[
\implies u' \in B(u, r_1), v' \in B(v, r_2) \implies u' \in U, v' \in V \implies (u', v') \in U \times V.
\]
This completes the proof.

(ii) To show \(U \times V\) is closed in \(X \times Y\) we show the complement of this set is open. We compute
\[
X \times Y \setminus (U \times V) = (X \setminus U) \times Y \cup X \times (Y \setminus V).
\]
If \(U\) is closed in \(X\) and \(V\) is closed in \(Y\) then \(X \setminus U\) is open in \(X\) and \(Y \setminus V\) is open in \(Y\). By (i), it follows that \((X \setminus U) \times Y\) and \(X \times (Y \setminus V)\) are open in \(X \times Y\), hence so is their union. This completes the proof.
Problem 2. [10 points.]

Let \((X, d)\) be a metric space, and let \(A, B, Y \subset X\) be subsets. True or false? If true, provide a careful justification. If false, provide a counterexample and correct the statement.

(i) \((A \cap B)' = A' \cap B'\)
(ii) \((A \cap B)^o = A^o \cap B^o\)

Solution:

(i) False. Take \(A = \{\frac{1}{n} : n \geq 1\}\) and \(B = \{-\frac{1}{n} : n \geq 1\}\). Clearly, \(A' = \{0\}\) and \(B' = \{0\}\) so \(A' \cap B' = \{0\}\). However \(A \cap B = \emptyset\) hence \((A \cap B)' \neq A' \cap B'\). For the corrected statement, we use that if \(E \subset F\) then \(E' \subset F'\) as shown in class. Then

\[
A \cap B \subset A \implies (A \cap B)' \subset A'
\]
\[
A \cap B \subset B \implies (A \cap B)' \subset B'
\]

hence

\((A \cap B)' \subset A' \cap B'\).

(ii) True. We show this by double inclusion:

\((A \cap B)^o \subset A^o \cap B^o\) and \(A^o \cap B^o \subset (A \cap B)^o\).

In one direction, if \(x \in (A \cap B)^o\) it follows that there exists a ball \(B(x, r) \subset A \cap B\). Then \(B(x, r) \subset A\) and \(B(x, r) \subset B\) which proves that \(x\) is interior to both \(A\) and \(B\). Thus \(x \in A^o\) and \(x \in B^o\). Therefore \(x \in A^o \cap B^o\).

Conversely, if \(x \in A^o \cap B^o\) it follows \(x\) is interior point for both \(A\) and \(B\). Thus, there are balls \(B(x, r_1) \subset A\) and \(B(x, r_2) \subset B\). Let \(r = \min(r_1, r_2)\). Thus

\[
B(x, r) \subset B(x, r_1) \subset A, \quad B(x, r) \subset B(x, r_2) \subset B \implies B(x, r) \subset A \cap B.
\]

Thus \(x\) is interior point for \(A \cap B\) or \(x \in (A \cap B)^o\).
Problem 3. [10 points.]

Consider \( \mathcal{F} \) the set of functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( -1 \leq f(x) \leq 1 \) for all \( x \in \mathbb{R} \). Define

\[
d(f, g) = \sup \{|f(x) - g(x)| : x \in \mathbb{R}\}.
\]

Show that \((\mathcal{F}, d)\) is a metric space.

Solution: First, \( d(f, g) \) is well-defined. The set of values

\[
|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq 1 + 1 = 2
\]

is bounded above, hence it makes sense to speak about the supremum.

We check that the axioms of a metric space are satisfied:

(i) Note that \( d(f, g) \geq 0 \) since \( |f(x) - g(x)| \geq 0 \) for all \( x \). Furthermore, \( d(f, g) = 0 \) means that 0 is an upper bound for \( |f(x) - g(x)| \) which means \( |f(x) - g(x)| = 0 \) for all \( x \) hence \( f(x) = g(x) \) for all \( x \). Thus \( f = g \).

(ii) The equality \( d(f, g) = d(g, f) \) is clear.

(iii) We prove that

\[
d(f, g) \leq d(f, h) + d(h, g).
\]

Fix \( x \in \mathbb{R} \). By the triangle inequality, we have

\[
|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq \\
\sup\{|f(x) - h(x)| : x \in \mathbb{R}\} + \sup\{|h(x) - g(x)| : x \in \mathbb{R}\} = d(f, h) + d(h, g).
\]

Thus \( d(f, h) + d(h, g) \) is an upper bound for the set

\[
\{|f(x) - g(x)| : x \in \mathbb{R}\}
\]

hence

\[
\sup\{|f(x) - g(x)| : x \in \mathbb{R}\} \leq d(f, h) + d(h, g) \implies d(f, g) \leq d(f, h) + d(h, g).
\]
**Problem 4.** [10 points.]

Let \( A \) be an infinite set, \( B \) a set with at least 2 elements, and define

\[
\mathcal{F}(A, B) = \{ \text{functions } f : A \to B \}.
\]

Show that the set \( \mathcal{F}(A, B) \) is uncountable.

**Solution:** We first claim that \( A \) contains a countable subset \( X \). Indeed, since \( A \) is infinite, we can define inductively a sequence of distinct elements

\[
\{ x_1, x_2, \ldots, x_n, \ldots \} \subset A.
\]

To this end, first pick \( x_1 \in A \) arbitrary, next pick \( x_2 \in A \setminus \{ x_1 \} \), and then continue: having picked \( x_1, \ldots, x_n \), we pick \( x_{n+1} \in A \setminus \{ x_1, \ldots, x_n \} \) which is possible since \( A \setminus \{ x_1, \ldots, x_n \} \neq \emptyset \) as \( A \) is infinite. Having defined the sequence, we can then partition

\[
A = X \cup Y
\]

where \( X = \{ x_1, x_2, \ldots, x_n, \ldots \} \) is countable, and \( Y = A \setminus X \) may be possibly empty.

Next, let \( b_1, b_2 \in B \) two distinct elements. We consider \( \mathcal{P} \) the set of all sequences whose terms equal either \( b_1 \) or \( b_2 \). We have seen in class that \( \mathcal{P} \) is uncountable.

We will define a subset \( Q \subset \mathcal{F}(A, B) \) which is in one-to-one correspondence with \( \mathcal{P} \). As a consequence, \( Q \) is also uncountable. If \( \mathcal{F}(A, B) \) were countable, then any of its subsets would be at most countable by a theorem in class. Since \( Q \) violates this, it follows that \( \mathcal{F}(A, B) \) must be uncountable, completing the proof.

It remains to define \( Q \) and the bijection \( F : \mathcal{P} \to \mathcal{Q} \). Fix \( s \in \mathcal{P} \) a sequence whose terms are equal to either \( b_1 \) or \( b_2 \). To the sequence \( s \) we associate a function \( f_s : A \to B \) as follows:

\[
\begin{cases}
\text{the } n\text{th term of the sequence } s & \text{if } a = x_n \\
b_1 & \text{if } a \in Y.
\end{cases}
\]

The association

\[
F : \mathcal{P} \ni s \mapsto f_s \in \mathcal{F}(A, B), \quad F(s) = f_s
\]

is clearly injective: indeed, if \( s \) and \( s' \) are two different sequences, they must differ in the \( n\text{th} \) term for some \( n \), then \( f_s(x_n) \neq f_{s'}(x_n) \), hence \( F(s) \neq F(s') \). It suffices to let

\[
Q = F(\mathcal{P}) \subset \mathcal{F}(A, B).
\]

Then \( F : \mathcal{P} \to \mathcal{Q} \) is injective onto its image, hence bijective.