Math 140A: Midterm 2
Foundations of Real Analysis

• You have 50 minutes.
• No books and notes are allowed.
• You may quote any result stated in the textbook or in class.
• You may not use homework problems (without proof) in your solutions.
1. (10 points)

(a) (5 points) Let \( \{a_n\} \) be a sequence of positive real numbers such that \( \lim_{n \to \infty} a_n = 0 \).

Prove that \( \lim_{n \to \infty} a_n^p = 0 \), for any \( p > 0 \).

(b) (5 points) Let \( \{a_n\} \), \( \{b_n\} \) and \( \{c_n\} \) be sequences of real numbers such that \( a_n \leq b_n \leq c_n \), for all \( n \geq 1 \). Assume that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = l \).

Prove that \( \lim_{n \to \infty} b_n = l \).
2. (10 points) Let \( \{a_n\} \) and \( \{b_n\} \) be bounded sequences of positive real numbers. Prove that 
\[
\limsup_{n \to \infty} (a_n b_n) \leq (\limsup_{n \to \infty} a_n)(\limsup_{n \to \infty} b_n).
\]
3. Let \( \{a_n\} \) be a sequence of real numbers such that \( \lim_{n \to \infty} a_n = 0 \). Define a new sequence \( \{b_n\} \) by letting \( b_n = a_n - a_{n+1} \), for every \( n \geq 1 \).

Prove that the series \( \sum_{n=1}^{\infty} b_n \) is convergent and \( \sum_{n=1}^{\infty} b_n = a_1 \).
4. (10 points) Let $a$ be a real number.

(a) (5 points) Prove that if $a \geq 1$, then the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^!}{(a+1)(a+2)\ldots(a+n)}$ is convergent.

(b) (5 points) Prove that if $a \geq 2$, then the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^!}{(a+1)(a+2)\ldots(a+n)}$ is absolutely convergent.
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