
Problem 1. Let $a_1, \ldots, a_5$ be pairwise distinct constants. Find the singularities of the projective hyperelliptic curve of genus 2:

$$y^2z^3 = (x - a_1z) \ldots (x - a_5z).$$

Answer: Let

$$f(x, y, z) = y^2z^3 - (x - a_1z) \ldots (x - a_5z).$$

Then $f$ is singular at $p$ if and only if

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Since

$$\frac{\partial f}{\partial y} = 2yz^3,$$

we see that if $p = [x : y : z]$ is a singular point then $y = 0$ or $z = 0$.

If $y = 0$, from $f(p) = 0$ we obtain $x = a_i$ for some $i$. Because we are free to scale the coordinates in $\mathbb{P}^2$, $p = [a_i : 0 : 1]$. We compute

$$\frac{\partial f}{\partial x} = -(x - a_2z)(x - a_3z)(x - a_4z)(x - a_5z) - (x - a_1z)(x - a_3z)(x - a_4z)(x - a_5z)$$

$$- (x - a_1z)(x - a_2z)(x - a_4z)(x - a_5z) - (x - a_1z)(x - a_2z)(x - a_3z)(x - a_5z)$$

$$- (x - a_1z)(x - a_2z)(x - a_3z)(x - a_4z).$$

Similarly,

$$\frac{\partial f}{\partial z} = 3y^2z^2 + a_1(x - a_2z)(x - a_3z)(x - a_4z)(x - a_5z) + a_2(x - a_1z)(x - a_3z)(x - a_4z)(x - a_5z)$$

$$+ a_3(x - a_1z)(x - a_2z)(x - a_4z)(x - a_5z) + a_4(x - a_1z)(x - a_2z)(x - a_3z)(x - a_5z)$$

$$+ a_5(x - a_1z)(x - a_2z)(x - a_3z)(x - a_4z).$$

Thus

$$\frac{\partial f}{\partial x}(p) = \prod_{j \neq i} (a_j - a_i) \neq 0$$

since $a_i$ are distinct. Thus $[a_i : 0 : 1]$ are not singular points.

If $z = 0$, $f(x, y, z) = x^3 = 0$, which is only possible when $x = 0$. Then $p = [0 : 1 : 0]$. The formulas above show

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Thus $[0 : 1 : 0]$ is the only singular point. \( \square \)

Problem 2. Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $d$ cut out by the homogeneous equations

$$f_1 = \ldots = f_r = 0.$$

We say that $X$ is singular at $p$ if the rank of the $r \times (n + 1)$ Jacobi matrix of partial derivatives

$$\left( \frac{\partial f_i}{\partial x_j}(p) \right), \ 1 \leq i \leq r, \ 0 \leq j \leq n,$$

strictly less than $n - d$.

Check that the twisted cubic

$$X = \{(x_0 : x_1 : x_2 : x_3) : x_1^2 - x_0x_2 = x_2^2 - x_1x_3 = x_0x_3 - x_1x_2 = 0\}$$

is nonsingular.
The Jacobian matrix is:
\[ \begin{pmatrix} a & 3 \\ 2 & x_1 \\ 0 & -2x_3 \\ x_3 & -x_2 \end{pmatrix} \]

Because the Jacobian has rank less than 2, we conclude that X has rank 0.

\[ J(\mathbf{p}) = \begin{pmatrix} -a & 2 & 0 \\ 0 & -2b & -a & 0 \\ b & 2 & -a & 0 \\ 2 & -a & -a & 0 \end{pmatrix} \]

If \( \mathbf{p} \) is a singular point, the submatrix

\[ \begin{vmatrix} -a & 2 \\ 0 & -2b \end{vmatrix} = 0. \]

because the Jacobian has rank less than 3 - 1 = 2. That implies either \( a = 0 \) or \( b = 0 \). Suppose \( a = 0 \). Suppose \( b = 0 \). Then the matrix

\[ J(\mathbf{p}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \]

has rank 2. Therefore \( \mathbf{p} \) is not a singular point. Similarly \( \mathbf{p} \) is not a singular point. We conclude X is not singular as desired.

\[ \square \]

**Problem 3.**

(i) Prove that if \( C = Z(f) \) and \( D = Z(g) \) are curves in \( \mathbb{P}^2 \), then

\[ \text{Sing } (C \cup D) = \text{Sing } (C) \cup \text{Sing } (D) \cup (C \cap D). \]

To prove this, note first that \( C \cup D = Z(fg) \).

(ii) Deduce that a nonsingular curve \( Z(F) \) in \( \mathbb{P}^2 \) is irreducible. Indeed, argue that if \( F \) has at least two distinct factors \( f \) and \( g \), then \( f \) and \( g \) must be homogeneous, and thus \( f \) and \( g \) must have at least a common zero which is a singular point for \( Z(F) \). Is this statement true for affine plane curves?

**Answer:**

(i) We first show that

\[ \text{Sing } (C) \cup \text{Sing } (D) \cup (C \cap D) \subset \text{Sing } (C \cup D). \]

If \( \mathbf{p} \in \text{Sing } (C) \), we have

\[ f(\mathbf{p}) = \frac{\partial f}{\partial x_i}(\mathbf{p}) = 0. \]

Under these assumptions,

\[ (fg)(\mathbf{p}) = 0, \quad \frac{\partial (fg)}{\partial x_i}(\mathbf{p}) = f(\mathbf{p}) \frac{\partial g}{\partial x_i}(\mathbf{p}) + g(\mathbf{p}) \frac{\partial f}{\partial x_i}(\mathbf{p}) = 0. \]

So \( \mathbf{p} \in \text{Sing } (C \cup D) \). Similarly, if \( \mathbf{p} \in \text{Sing } (D) \), then \( \mathbf{p} \in \text{Sing } (C \cup D) \).

If \( \mathbf{p} \in C \cap D \), then \( f(\mathbf{p}) = g(\mathbf{p}) = 0 \). Then \( (fg)(\mathbf{p}) = 0 \) and the derivatives are

\[ \frac{\partial fg}{\partial x_i}(\mathbf{p}) = f(\mathbf{p}) \frac{\partial g}{\partial x_i}(\mathbf{p}) + g(\mathbf{p}) \frac{\partial f}{\partial x_i}(\mathbf{p}) = 0. \]
Therefore \( p \in \text{Sing} (C \cup D) \).

Conversely, we show

\[
\text{Sing} (C \cup D) \subset \text{Sing} (C) \cup \text{Sing} (D) \cup (C \cap D).
\]

If \( p \in \text{Sing} (C \cup D) \), at least \( p \in C \) or \( D \). If \( p \in C \), \( f(p) = 0 \).

Then this implies \( g(p) = 0 \) or else \( \frac{\partial f}{\partial x_i}(p) = 0 \) for all \( i \). Thus, either \( p \in (C \cap D) \) or \( p \in \text{Sing}(C) \).

Putting everything together, we have shown that

\[
\text{Sing} (C \cup D) = \text{Sing} (C) \cup \text{Sing} (D) \cup (C \cap D).
\]

(ii) If a curve \( C \) is the union of two curves \( C_1 \) and \( C_2 \) in \( \mathbb{P}^2 \), \( C_1 \) and \( C_2 \) must intersect in \( \mathbb{P}^2 \).

From (i) we know their intersection points are singularities of \( C \).

This doesn’t apply to affine curves. For example, \( x(x - 1) = 0 \subset \mathbb{A}^2 \) is the reducible union of two distinct lines and it is smooth.

**Problem 4.** Show that a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is non-singular:

(i) For any hypersurface \( Z(f) \subset \mathbb{P}^n \) of degree \( d \), view the coefficients of \( f \) as a point \( p_f \) in a large dimensional projective space \( \mathbb{P}^N \) (This projective space is called the moduli space of degree \( d \) hypersurfaces). Let \( X = \{(f, p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f \} \).

Show that \( X \) is a projective algebraic set in \( \mathbb{P}^N \times \mathbb{P}^n \).

(ii) Conclude that the image \( \pi(X) \) of \( X \) under the projection onto \( \mathbb{P}^N \) is a projective algebraic set. What is \( \pi(X) \)? Conclude that the subset of \( \mathbb{P}^n \) corresponding to smooth hypersurfaces is open and nonempty.

**Answer:**

(i) Let

\[
f = \sum_I a_I X^I,
\]

where \( I \) is a multi-index. Then \( a_I \) will be the coordinates of the point \( p_f \) in \( \mathbb{P}^N \).

Let \( p = [x_0 : x_1 : \ldots : x_n] \), the condition \( f \) is singular at \( p \) is equivalent to

\[
f(p) = \frac{\partial f}{\partial X_i}(p) = 0 \text{ for all } i \text{ and }
\]

\[
\sum_I a_I x^I = \frac{\partial(\sum_I a_I X^I)}{\partial X_i}(p) = 0.
\]

Let \( a_I \) vary in \( \mathbb{P}^N \) and \( x \) vary in \( \mathbb{P}^n \), the equations above can be viewed as equations of \( a_I \) and \( x \) in \( \mathbb{P}^N \times \mathbb{P}^n \) which are bi-homogeneous in the variables. Therefore \( X \) is projective algebraic.

(ii) As shown in class, the projection \( \pi \) is a closed map e.g. \( \pi(X) \) is a projective algebraic set. Note that

\[
\pi(X) = \{f \mid f \text{ is a nonsingular homogeneous degree } d \text{ polynomial}\}
\]

The complement of \( \pi(X) \) is open, therefore nonsingular homogeneous degree \( d \) polynomial form an open set in the moduli space.
To show non-emptyness, observe that

$$f = X^d_0 + \ldots + X^d_n$$

is a homogeneous polynomial without singularities. Indeed, all derivatives of $f$ are $dX_i^{d-1}$ which do not have a common vanishing in $\mathbb{P}^n$.

\[\square\]

**Problem 5.** Solve problem 6.6 part (a) in the textbook.

**Answer:** The curve $\sigma^{-1}(C_n)$ is given by

$$f(x, ux) = (ux)^2 - x^{2n+1} = u^2 - x^{2n-1}$$

is the union of $2l$ and $C_{n-1}$. Continuing the process until $n = 1$, $C_1$ is given by $y^2 - x = 0$ is nonsingular, thus $X_n$ resolves after a chain of $n$ blow-ups.

\[\square\]

**Problem 6.** Consider the singular plane curves $Z$ and $W$ given by the equations

$$y^2 - x^2(x + 1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain briefly why $Z$ and $W$ are not isomorphic. Explain that $(0, 0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0, 0)$ for $Z$ and $W$? Sketch (the real points of) $Z$ and $W$. Do $Z$ and $W$ look alike near the origin?

(ii) Show that there are formal power series

$$\tilde{x} = f_1 + f_2 + f_3 + \ldots$$

$$\tilde{y} = g_1 + g_2 + g_3 + \ldots$$

in the variables $x$ and $y$ such that the equation of $Z$ becomes

$$\tilde{x}\tilde{y} = 0.$$ 

(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

**Answer:**

(i) First, $Z$ and $W$ are not isomorphic because $Z$ is irreducible and $W$ is reducible.

Now, $Z$ is defined by $y^2 - x^2(x + 1) = 0$. The tangent lines can be found by considering the lowest degree terms $y^2 - x^2$. This factors as $(y - x)(y + x)$. So the 2 tangent lines are $y - x = 0$ and $y + x = 0$.

Similarly, $W = (xy = 0)$ is the union of two lines. The tangent lines are just these two lines $x = 0$ and $y = 0$.

(ii) We hope to find $f_i$ and $g_i$ such that

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).$$

First, compare the degree 2 terms, then

$$f_1 g_1 = y^2 - x^2.$$

Hence, we can take

$$f_1 = y - x \text{ and } g_1 = y + x.$$

Comparing the degree 3 terms we have

$$-x^3 = (y - x)g_2 + (y + x)f_2.$$

Let $g_2 = x^2/2$ and $f_2 = -x^2/2$ will work.

Suppose we have found $f_i$ and $g_i$ for $1 \leq i \leq d - 1$ and

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1)$$
up to degree \(d\). Comparing degree \(d + 1\) terms, we have:

\[ f_1 g_d + f_2 g_{d-1} + \cdots + f_d g_1 = 0. \]

Now only \(f_d\) and \(g_d\) are unknown, others are fixed, we can rearrange the equation:

\[(y - x)g_d + (y + x)f_d = -(f_2 g_{d-1} + \cdots + f_{d-1} g_2).\]

Notice that

\[ f_2 g_{d-1} + \cdots + f_{d-1} g_2 \]

is a homogeneous polynomial of degree \(d + 1\). Let

\[-(f_2 g_{d-1} + \cdots + f_{d-1} g_2) = ax^d + yR(x, y).\]

Isolating \(x^{d+1}\) and dividing the remaining term by \(y\) to obtain \(R\), then we need

\[ x(f_d - g_d) + y(f_d + g_d) = ax^{d+1} + yR(x, y) \]

This is possible by letting

\[ f_d = \frac{1}{2} \left( \frac{a}{2} x^d + R(x, y) \right) \quad \text{and} \quad g_d = \frac{1}{2} \left( -\frac{a}{2} x^d + R(x, y) \right) \]

Therefore we can find \(f_d\) and \(g_d\) and

\[(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1) \]

up to degree \(d + 1\). Inductively, there exist

\[ \tilde{x} = f_1 + f_2 + f_3 + \ldots \]
\[ \tilde{y} = g_1 + g_2 + g_3 + \ldots \]

such that

\[(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).\]

(iii) Suppose \(C = (H = 0)\) is a curve which has a ordinary double point, we can change coordinates to assume that the singularity is at the origin. Because \(C\) has a double point at the origin,

\[ H(x, y) = H_2(x, y) + H_3(x, y) + \cdots \quad \deg H_i = k \]

where \(H_2\) is a homogeneous polynomial of degree 2, with distinct factors

\[ H_2 = (ax + by)(cx + dy). \]

Change coordinates so that

\[ x' = ax + by, y' = cx + dy. \]

In the new coordinates,

\[ H = x'y' + H_3 + H_4 + \ldots. \]

By the same method as in (ii), we inductively find

\[ \tilde{x} = x' + f_2 + f_3 + \ldots \]
\[ \tilde{y} = y' + g_2 + g_3 + \ldots \]

such that

\[ H = \tilde{x}\tilde{y}. \]