Problem 1. Let $X$ and $Y$ be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively.

(i) Construct the product prevariety $X \times Y$ by gluing the affine varieties $U_i \times V_j$ together.

(ii) Show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$
satisfying the usual universal property for products: given morphisms $f : Z \to X$ and $g : Z \to Y$ from any prevariety $Z$, there is a unique morphism $h : Z \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.

(iii) Show that if $X$ and $Y$ are varieties then $X \times Y$ is also a variety.

Answer: 

(i) We have already seen that products of affine algebraic sets are well defined. Thus, by restriction, it follows that products of quasi-affine sets are well defined. Let

$$U_{ij} = U_i \cap U_j \text{ and } V_{ij} = V_i \cap V_j.$$ 

We glue the products of affine algebraic sets $U_i \times V_k$ and $U_j \times V_l$ along the products of quasiaffine sets $U_{ij} \times V_{kl}$. It is clear that the cocycle condition on triple overlaps for the gluing maps is satisfied.

(ii) It is clear that the projections $\pi_X$ and $\pi_Y$ are morphisms. Assume $f, g, Z$ are given. Set theoretically, we define

$$h(z) = (f(z), g(z)) : Z \to X \times Y.$$ 

We need to check that $h$ is a morphism. Clearly $h$ is continuous. We check that if $\phi$ is a regular function on some open set $W \subset X \times Y$ then $h^* \phi$ is regular on $h^{-1}W$. Let $Z_{ij}$ be the preimages of $U_i \times V_j$ under the morphism $h$. Clearly, $Z_{ij}$ is open, hence it is a prevariety, hence it can be covered by affines $Z_{ij}^l$. Since the restriction $h : W_{ij}^l \to U_i \times V_j$ is a morphism of affine sets, it must correspond to a polynomial map. Thus $h^* \phi$ is clearly regular on $h^{-1}(W) \cap Z_{ij}^l$, e.g. it gives rise to a section of the sheaf $\mathcal{O}_Z(h^{-1}(W) \cap Z_{ij})$. These sections agree on overlaps and give a regular section of $\mathcal{O}_X(h^{-1}(W))$ which must be equal to $h^* \phi$. This proves the claim.

It follows from the universal property that $X \times Y$ is unique up to isomorphism.

(iii) If $h_1, h_2 : Z \to X \times Y$, we obtain morphisms

$$f_1, f_2 : Z \to X, \quad g_1, g_2 : Z \to Y.$$ 

We have

$$\{z \in Z : h_1(z) = h_2(z)\} = \{z \in Z : f_1(z) = f_2(z)\} \cap \{z \in Z : g_1(z) = g_2(z)\}.$$ 

The latter two sets are closed since $X$ and $Y$ are varieties, hence their intersection is also closed as well, showing that $X \times Y$ is a variety.

Problem 2. Let $X$ be a prevariety. Consider pairs $(U, f)$ where $U$ is an open subset of $X$ and $f \in \mathcal{O}_X(U)$ a regular function on $U$. We call two such pairs $(U, f)$ and $(U', f')$ equivalent if there is an open subset $V$ in $X$ with $V \subset U \cap U'$ such that

$$f|_V = f'|_V.$$
(i) Show that this defines an equivalence relation.
(ii) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the field of rational functions on $X$ and denoted $K(X)$.
(iii) If $X$ is an affine variety, show that $K(X)$ is just the field of rational functions as defined in class.

Answer: (i) Checking that we have defined an equivalence relation is straightforward. For instance, to verify transitivity, let $(U, f)$, $(U', f')$ and $(U'', f'')$ such that $f|_V = f'|_V$ and $f'|_W = f''|_W$ for some open sets $V \subset U \cap U'$ and $W \subset U' \cap U''$.

We conclude $f|_T = f''|_T$ where $T = V \cap W \subset U \cap U''$.

Clearly, $T$ is nonempty since $V, W$ are open dense in the irreducible set $X$.

(ii) It is easy to see that addition and multiplication pass through the equivalence relation. We will check that non-zero elements have inverses. Let $(f, U)$ be a non-zero element. In particular $Z(f) \neq U$. Consider the function $(1/f, U \setminus Z(f))$ which is regular on the nonempty set $U \setminus Z(f)$. This is the inverse of $(f, U)$.

(iii) If $f$ is a rational function as defined in class, we take $U = \text{Domain}(f)$ which is open in $X$. We obtain an equivalence class $(U, f)$ as in the problem. Conversely, if $(f, U)$ is an equivalence class as in the problem set, then $f$ is regular on $U$ hence by definition, it must be a rational function on $X$. This assignment $(f, U) \mapsto f$ passes through the equivalence relation. Indeed, if $(f', U')$ is another representative of the equivalence class, then $f = f'$ on some open set $V$. Write $f = g/h$ and $f' = g'/h'$. We show that $g/h$ and $g'/h'$ define the same element of the function field. Indeed,

$$\frac{g}{h} = \frac{g'}{h'} \text{ on } V \implies gh' - g'h = 0 \text{ on } V \implies gh' - g'h = 0 \text{ on } X$$

$$\implies \frac{g}{h} = \frac{g'}{h'} \text{ as elements of } K(X).$$

For the third implication, we used that $V$ is dense in $X$.

\[ \square \]

Problem 3. (i) Show that every isomorphism $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is of the form $f(x) = ax + b$.
(ii) Show that every isomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is of the form $f(x) = \frac{ax + b}{cx + d}$ for some $a, b, c, d \in k$, where $x$ is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.
(iii) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

Answer: (i) Clearly, a morphism $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ must be a polynomial map. If $f$ has degree $d$, the equation $f(x) = 0$ will have $d$ roots. Therefore, $d = 1$ hence $f(x) = ax + b$. 

(ii) As shown above, every isomorphism is of the form $f(x) = \frac{ax + b}{cx + d}$.

(iii) The assignment $(P_1, P_2, P_3) \mapsto (Q_1, Q_2, Q_3)$ passes through the equivalence relation. Indeed, if $(P_1', P_2', P_3')$ is another representative of the equivalence class, then $f = f'$ on some open set $V$. Write $f = g/h$ and $f' = g'/h'$. We show that $g/h$ and $g'/h'$ define the same element of the function field. Indeed,
(ii) Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \). If \( f(\infty) = \infty \), then \( f \) maps \( \mathbb{A}^1 \to \mathbb{A}^1 \) and must have the form \( f(x) = ax + b \). Otherwise, if \( f(\infty) = \lambda \), let \( g(x) = \frac{1}{x-\lambda} \). Then \( g \circ f \) is an isomorphism which maps \( \infty \) to itself hence it must have the form

\[
g \circ f = cx + d.
\]

This shows

\[
f(x) = \lambda + \frac{1}{cx + d} = ax + b
\]

for

\[
a = c\lambda, b = d\lambda + 1.
\]

(iii) By transitivity, it suffices to assume \( P_1 = 0, P_2 = 1, P_3 = \infty \). Also, let \( Q_0 = \lambda, Q_1 = \mu, Q_2 = \nu \). Assume that none of the greek letters is \( \infty \). We require

\[
\frac{b}{d} = \lambda, \frac{a + b}{c + d} = \mu, \frac{a}{c} = \nu.
\]

We may take

\[
a = \nu(\mu - \lambda), \ b = \lambda(\nu - \mu), \ c = \mu - \lambda, \ d = \nu - \mu.
\]

If one of the greek letters is \( \infty \), say \( \nu = \infty \), and \( \mu, \lambda \neq 0 \), we may apply the transformation \( x \to \frac{x}{\nu} \) to reduce to the case we have already studied.

If by contrast, \( \lambda = 0 \), then the automorphism \( f(x) = \mu x \) sends the \( P \)'s to the \( Q \)'s. If \( \mu = 0 \), then we may take

\[
f(x) = \frac{x - 1}{\lambda}.
\]

To prove uniqueness, assume that two morphisms \( f_1, f_2 \) have been constructed. The inverse \( f = f_1 \circ f_2^{-1} \) has 3 fixed points \( P_1, P_2, P_3 \). We may assume that these fixed points are \( 0, 1, \infty \). Since

\[
f(x) = \frac{ax + b}{cx + d}
\]

and

\[
f(0) = 0, f(1) = 1, f(\infty) = \infty
\]

we conclude

\[
b = 0, a + b = c + d, c = 0 \implies f = \text{id} \implies f_1 = f_2.
\]

Problem 4. Let \( X \subset \mathbb{P}^2 \) be a projective variety. A morphism \( f : \mathbb{P}^1 \to X \) is a polynomial map

\[
f([x : y]) = (f_0([x : y]), f_1([x : y]), f_2([x : y])),
\]

where \( f_0, f_1, f_2 \) are homogeneous polynomials of the same degree, such that \( f(\mathbb{P}^1) \subset X \).

Prove the following facts about lines and conics in projective plane:

(i) For any line \( L \subset \mathbb{P}^2 \), there is a bijective morphism

\[
f : \mathbb{P}^1 \to L.
\]

(ii) For any irreducible conic \( C \subset \mathbb{P}^2 \), there is a bijective morphism

\[
f : \mathbb{P}^1 \to C.
\]

You may wish to change coordinates so that your conic has a convenient expression.
(iii) Consider the elliptic curve $E_\lambda \subset \mathbb{P}^2$:
\[
y^2z = x(x - z)(x - \lambda z).
\]
Show that there are no nonconstant morphisms
\[
\mathbb{P}^1 \to E_\lambda \subset \mathbb{P}^2.
\]
Therefore, elliptic curves are not rational curves.

Answer: (i) Let $L$ be the line $\alpha x + \beta y + \gamma z = 0$. We may assume that $\gamma \neq 0$, eventually relabeling the coordinates if necessary. Set
\[
f : \mathbb{P}^1 \to L, \quad [x : y] \to \left[x : y : -\frac{\alpha}{\gamma}x - \frac{\beta}{\gamma}y\right],
\]
and
\[
g : L \to \mathbb{P}^1 \quad [x : y : z] \to [x : y].
\]
It is easy to see that both $f$ and $g$ are well defined inverse isomorphisms.

(ii) Changing coordinates, we may assume the conic $C$ is given by the equation
\[
xz = y^2.
\]
Set
\[
f : \mathbb{P}^1 \to C, \quad [s : t] \to [s^2 : st : t^2].
\]
It is easy to check that
\[
g : C \to \mathbb{P}^1, \quad g([x : y : z]) = \begin{cases} [x : y] & \text{if } (x, y) \neq (0, 0) \\ [y : z] & \text{if } (y, z) \neq (0, 0) \end{cases}
\]
is an inverse morphism of $f$.

(iii) Assume that $F : \mathbb{P}^1 \to E_\lambda \subset \mathbb{P}^2$ is a morphism with
\[
F([x : y]) = [f_0([x : y]) : f_1([x : y]) : f_2([x : y])],
\]
where
\[
f_1^2f_2 = f_0(f_0 - f_1)(f_0 - \lambda f_1).
\]
If $f_2 \equiv 0$, it follows from the equation above that $f_0 = 0$, hence $F$ is the constant $[0 : 1 : 0]$. Let us assume $f_2 \neq 0$. Define
\[
f(t) = \left(\frac{f_0(t, 1)}{f_2(t, 1)}, \frac{f_1(t, 1)}{f_2(t, 1)}\right).
\]
It is clear that $f$ defines a rational map
\[
f : \mathbb{A}^1 \dashrightarrow E_\lambda
\]
which must be constant by the previous pset. This implies that $f_0(t, 1)/f_2(t, 1)$ and $f_1(t, 1)/f_2(t, 1)$ are constants. It follows that all points $[t : 1]$ take the same value under $F$. Similarly, $F$ sends all $[1 : t]$ to the same value, and therefore $F$ must be constant.
Problem 5. Part A. Show that a line and an irreducible conic in \( \mathbb{P}^2 \) cannot intersect in 3 points.

Part B.

(i) Four points in \( \mathbb{P}^2 \) are said to be in general position if no three are collinear (i.e. lie on
a projective line in the projective plane). Show that if \( p_1, \ldots, p_4 \) are points in general
position, there exists a linear change of coordinates
\[
T : \mathbb{P}^2 \to \mathbb{P}^2
\]
with
\[
T([1 : 0 : 0]) = p_1, \quad T([0 : 1 : 0]) = p_2, \quad T([0 : 0 : 1]) = p_3, \quad T([1 : 1 : 1]) = p_4.
\]

(ii) Given five distinct points in \( \mathbb{P}^2 \), no three of which are collinear, show that there is an
unique irreducible projective conic passing through all five points. You may want to use
part (i) to assume that four of the points are \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]\).

(iii) Deduce that two distinct irreducible conics in \( \mathbb{P}^2 \) cannot intersect in 5 points. (We will
see later that they intersect in exactly 4 points counted with multiplicity.)

Answer: Part A. Changing coordinates, we may assume that the conic \( C \) is
\[
xz = y^2.
\]

Let \( L \) be the line
\[
\alpha x + \beta y + \gamma z = 0.
\]
If \( \beta = 0 \), it is clear that \( C \) and \( L \) intersect at the points
\[
[-\gamma : \pm \sqrt{\alpha \gamma} : \alpha].
\]
If \( \beta \neq 0 \), then
\[
y = -\frac{\alpha}{\beta} x - \frac{\gamma}{\beta} z
\]
which gives
\[
xz = \frac{1}{\beta^2} (\alpha x + \gamma z)^2.
\]
If \( x = 0 \), then \( y = z = 0 \) which is not allowed. Therefore, we may assume \( x \neq 0 \).
Dividing by \( x^2 \) we obtain the quadratic equation in \( \frac{z}{x} \):
\[
\frac{\gamma^2}{\beta^2} \left( \frac{z}{x} \right)^2 + \left( \frac{2 \alpha \gamma}{\beta^2} - 1 \right) \frac{z}{x} + \frac{\alpha^2}{\beta^2} = 0.
\]
Letting \( \lambda_1, \lambda_2 \) be the solutions of this equation, we see that the intersection points are
\[
\left[ 1 : -\frac{\alpha}{\beta} - \frac{\gamma}{\beta} \lambda_i : \lambda_i \right].
\]

Part B.

(i) Let \( p_i = [a_i : b_i : c_i] \) for \( 1 \leq i \leq 4 \). Define
\[
A = \begin{pmatrix}
\alpha a_1 & \alpha b_1 & \alpha c_1 \\
\beta a_2 & \beta b_2 & \beta c_2 \\
\gamma a_3 & \gamma b_3 & \gamma c_3
\end{pmatrix},
\]
where \( \alpha, \beta, \gamma \) will be specified later. In fact, we will require that \( \alpha, \beta, \gamma \) solve the system
\[
\alpha a_1 + \beta b_1 + \gamma c_1 = a_4,
\]
\[ \alpha a_2 + \beta b_2 + \gamma c_2 = b_4, \]
\[ \alpha a_3 + \beta b_3 + \gamma c_3 = c_4. \]

A solution exists since the matrix of coefficients
\[
B = \begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
\end{pmatrix}
\]
is invertible. Indeed, the rows of \( B \) are independent. Otherwise, a nontrivial linear relation between the rows would give a line on which the points \( p_1, p_2, p_3 \) lie. Thus \( B \) is invertible. Now, the system above has the solution
\[
\begin{bmatrix}
  \alpha \\
  \beta \\
  \gamma
\end{bmatrix} = B^{-1} \begin{bmatrix}
  a_4 \\
  b_4 \\
  c_4
\end{bmatrix}.
\]

Note that the same argument shows that \( A \) is invertible. Let
\[
S : \mathbb{P}^2 \rightarrow \mathbb{P}^2
\]
be the linear transformation defined by \( A \). Then \( S \) is invertible. A direct computation shows
\[
S([1 : 0 : 0]) = p_1, S([0 : 1 : 0]) = p_2, S([0 : 0 : 1]) = p_3, S([1 : 1 : 1]) = p_4.
\]
The proof is completed letting \( T \) be the inverse of \( S \).

(ii) After a linear change of coordinates, we may assume that the five points are \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1] \text{ and } [u : v : w] \). The equation \( f(x, y, z) \) of any conic passing through the first three points can’t contain \( x^2, y^2, z^2 \), so
\[
f(x, y, z) = ayz + bxz + cxy.
\]
The remaining two points impose the conditions
\[
a + b + c = avw + buw + cuv = 0.
\]
Letting
\[
A = \begin{pmatrix}
  1 & 1 & 1 \\
  vw & uw & uv
\end{pmatrix},
\]
we see that \((a, b, c)\) must be in the null space of \( A \). The rank of \( A \) is 2 (the rank cannot be 1 since then the rows would be proportional, hence \( u = v = w \) which is not allowed). Therefore, the null space of this matrix is one dimensional, hence the conic passing through the 5 points is unique. The conic cannot be reducible since then it would be union of two lines. One of the lines would have to contain 3 points but that contradicts the general position assumption.

(iii) By Part A, a conic and a line cannot intersect in 3 points. Therefore, any 5 points on an irreducible conic are in general position. Now, the claim is evident by (ii).

\(\square\)

**Problem 6.** We will make the space of all lines in \( \mathbb{P}^n \) into a projective variety. We define a set-theoretic map
\[
\phi : \{\text{lines in } \mathbb{P}^n\} \rightarrow \mathbb{P}^N
\]
with
\[ N = \binom{n + 1}{2} - 1 \]
as follows. For every line \( L \subset \mathbb{P}^n \), choose two distinct points
\[ P = (a_0 \ldots a_n) \text{ and } Q = (b_0 \ldots b_n) \]
on \( L \) and define \( \phi(L) \) to be the point in \( \mathbb{P}^N \) whose homogeneous coordinates are the maximal minors of the matrix
\[
\begin{pmatrix}
a_0 & \ldots & a_n \\
b_0 & \ldots & b_n
\end{pmatrix}
\]
in any fixed order. Show that:

(i) The map \( \phi \) is well-defined and injective. The map \( \phi \) is called the Plucker embedding.

(ii) The image of \( \phi \) is a projective variety that has a finite cover by affine spaces \( \mathbb{A}^{2(n-1)} \). You may want to recall the Gaussian algorithm which brings almost any matrix as above into the form
\[
\begin{pmatrix}
1 & 0 & a'_2 & \ldots & a'_n \\
0 & 1 & b'_2 & \ldots & b'_n
\end{pmatrix}
\]

(iii) Show that \( G(1, 1) \) is a point, \( G(1, 2) = \mathbb{P}^2 \), and \( G(1, 3) \) is the zero locus of a quadratic equation in \( \mathbb{P}^5 \).

Answer: (i) Let \( e_0, \ldots, e_n \) be the standard basis of the vector space \( V = k^{n+1} \). A line \( L \) corresponds to a 2-dimensional subspace in \( k^{n+1} \), also denoted by \( L \). We claim that the map \( \phi \) can be described as follows. Picking a basis \( v, w \) for the subspace,
\[
\phi(L) = [v \wedge w] \in \mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^N.
\]
Indeed, if \( P = (a_0 : \ldots : a_n) \) and \( Q = (b_0 : \ldots : b_n) \) are two distinct points, then we may take
\[ v = \sum a_i e_i, \quad w = \sum b_i e_i. \]
Thus
\[ v \wedge w = \sum_{i,j} a_i b_j e_i \wedge e_j = \sum_{i<j} (a_i b_j - a_j b_i) e_i \wedge e_j. \]
Therefore, in the basis \( e_i \wedge e_j \), the coordinates are the \( 2 \times 2 \)-minors of the matrix in (i). This shows that \( \phi \) is well-defined.

To check injectivity, let \( L' \) be another line corresponding to a 2-dimensional subspace. If \( L' \cap L = 0 \), then pick a basis \( v_0, v_1, v_2, v_3 \) for \( L \oplus L' \) with
\[ v_0, v_1 \in L, v_2, v_3 \in L', \]
and extend it to a basis \( v_0, \ldots, v_n \) of \( V = k^{n+1} \). Thus \( v_i \wedge v_j \) for \( i < j \) is a basis for \( \Lambda^2 V \). In particular, \( v_0 \wedge v_1 \) and \( v_2 \wedge v_3 \) are not multiples, or equivalently
\[ \phi(L) \neq \phi(L'). \]
The same argument applies if \( L \) and \( L' \) have a 1 dimensional intersection.
(ii) To show projectivity, we will prove that
\[ \omega \in \Lambda^2 V \text{ splits as } \omega = v \land w \text{ if and only if } \omega \land \omega = 0. \]
In particular, if
\[ \omega = \sum \omega_{ij} e_i \land e_j, \]
then
\[ \omega \land \omega = \sum_{i<j,k<l} \omega_{ij} \omega_{kl} e_i \land e_j \land e_k \land e_l = \sum_{i<j<k<l} (\omega_{ij} \omega_{kl} - \omega_{ik} \omega_{jl} + \omega_{il} \omega_{jk}) e_i \land e_j \land e_k \land e_l \]
so the image of \( \phi \) is cut by the quadrics
\[ \omega_{ij} \omega_{kl} - \omega_{ik} \omega_{jl} + \omega_{il} \omega_{jk} = 0. \]
We now prove the claim. It is clear that if \( \omega = v \land w \) then \( \omega \land \omega = 0 \). Conversely, we will induct on \( n \), the base case \( n = 2 \) being clear. Let us write
\[ \omega = e_0 \land \eta + \omega' \]
where \( \omega', \eta \) do not contain the vector \( e_0 \). Thus
\[ 0 = \omega \land \omega = 2e_0 \land \eta \land \omega' + \omega' \land \omega'. \]
This implies that
\[ \omega' \land \omega' = 0 \]
and hence by induction
\[ \omega' = v \land w, \]
with \( v, w \) being in the subspace spanned by \( e_1, \ldots, e_n \). Also, we know
\[ e_0 \land \eta \land \omega' = 0 \implies \eta \land v \land w = 0. \]
This shows that \( \eta \) cannot be independent of \( v, w \) hence
\[ \eta = av + bw. \]
Collecting terms we find
\[ \omega = e_0 \land (av + bw) + v \land w = (v + be_0) \land (w + ae_0) \]
as claimed.
For the last part, note that one of the coordinates of \( \phi(L) \) must be non-zero. Without loss of generality let us assume it is the coordinate corresponding to \( e_0 \land e_1 \). This means that the first \( 2 \times 2 \) minor of the matrix
\[
\begin{pmatrix}
a_0 : & \ldots : & a_n \\
b_0 : & \ldots : & b_n
\end{pmatrix}
\]
is non-zero. The Gaussian algorithm brings this matrix into the form
\[
\begin{pmatrix}
1 & 0 & a'_2 & \ldots & a'_n \\
0 & 1 & b'_2 & \ldots & b'_n
\end{pmatrix},
\]
The association
\[ L \rightarrow (a'_2, \ldots, a'_n, b'_2, \ldots, b'_n) \in \Lambda^2(n-1) \]
shows how to cover the image of \( \phi \) by affine opens isomorphic to \( \Lambda^2(n-1) \).
(iii) All these statements are particular cases of what we proved in part (i). For instance, to see that $G(1, 3)$ is a quadric in $\mathbb{P}^5$ given by

$$x_0x_5 - x_1x_4 + x_2x_3 = 0.$$