Math 203, Problem Set 8. Due Friday, December 7.

Solve the following problems, and hand in solutions to three of them.

1. (Degree of the Segre embedding.) Show that the Segre embedding
\[ \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} \]
has degree \( \binom{n+m}{n} \).

2. (Arithmetic genus.) Let \( X \subset \mathbb{P}^n \) be a projective scheme with Hilbert polynomial \( \chi_X \). Define the arithmetic genus of \( X \) to be
\[ p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1). \]
(i) Show that the genus of \( \mathbb{P}^n \) is zero.
(ii) If \( X \) is a hypersurface of degree \( d \) in \( \mathbb{P}^n \), show that \( p_a(X) = \binom{d-1}{n} \). In particular, a cubic in \( \mathbb{P}^2 \) has genus 1.
(iii) If \( X \) is a complete intersection of two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) then
\[ p_a(X) = \frac{1}{2}ab(a + b - 4) + 1. \]
In particular, intersection of two quadrics in \( \mathbb{P}^3 \) has genus 1.

Remark: To compare (ii) and (iii), recall that a cubic in \( \mathbb{P}^2 \) is isomorphic to an intersection of two quadrics in \( \mathbb{P}^3 \) as shown in a previous homework.

3. (Enumerative geometry of lines.) Given four general lines in \( \mathbb{P}^3 \), show that there are exactly 2 lines which intersect all four of them.

Hint: Recall that the Grassmannian \( G(1, 3) \) is a quadric in \( \mathbb{P}^5 \) via the Plücker embedding.

Challenge: You may attempt to prove the following generalization:

Show that the number of lines in \( \mathbb{P}^n \) which intersect \( 2(n-1) \) fixed general codimension 2 linear hyperplanes equals the Catalan number
\[ C_n = \frac{1}{n} \binom{2n - 2}{n-1}. \]

4. (Varieties of minimal degree.) Let \( X \) be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree \( d \) and codimension \( c \) in \( \mathbb{P}^n \).

(i) (Intersecting \( X \) with hyperplanes to cut down the dimension), show inductively that
\[ d \geq c + 1. \]
(iii) Show that equality holds for rational normal curves in $\mathbb{P}^n$, and for the image $v(\mathbb{P}^2)$ of the Veronese embedding

$$v : \mathbb{P}^2 \to \mathbb{P}^5.$$ 

(iii) Can you classify the varieties of degree 2?

Remark: The del Pezzo-Bertini theorem classifies all varieties for which equality holds in (i).